M. Luptáčik: Mathematical Optimization and Economic Analysis

2009, Springer, New York-Dordrecht-Heidelberg-London, ISBN 978-0-387-89551-2

Pavel Doležel*

Economic theory is sometimes defined as a theory of allocating scarce resources for production. This allocation is studied in many ways, but nearly always finishes with a focus on its efficiency. Although efficiency itself can have several different definitions, the simplest question to be answered is always about "how to allocate inputs to obtain as much output as possible". This question is a typical question of mathematical optimization. Mathematical optimization is a very broad and interesting part of mathematics and a crucial part of almost any quantitative economic theory. It would be only slightly incorrect to state that economists deal with mathematical optimization more than most mathematicians. However, mathematical optimization is part of pure mathematics and is far more general than just its economic applications, although a significant part of optimization theory has been developed for economic applications. Optimization plays a role in many areas of human knowledge, including estimating parameters of statistical predictive models with impacts on medicine, physics, psychology, sociology, chemistry, genetics, biology, linguistics, informatics, cryptology, economics and many more.

Economics, however, introduces some specific problems that have historically played a greater role in forming optimization theory than any other of the listed areas. Trying to optimize is more fundamental to economics than to any other natural or social sciences. The reason might be in defining the optimization rule and the simplicity of the problem. In economics of production, the rule is simply the maximization of profit, but in medicine, for example, it can be difficult to define the optimization rule. In medicine two or more treatments are usually compared in terms of security and efficiency, but no treatment can be thought to be "globally" optimal as we never know whether there is another treatment, that is not yet known, which is superior to all the known treatments. The comparison itself is also an optimization method, but more or less trivial and not global.

Luptáčik's book is an interesting contribution to optimization methods used in economics and is a very well done overview of classical and recent optimization methods with applications in economics. The book is focused on the part of optimization called mathematical programming and does not deal with combinatorial optimization (optimization in networks, optimization on graphs, etc.) or stochastic optimization. The book is divided in two parts:

1. Single-objective optimization,
2. Multi-objective optimization.

The section on single-objective optimization deals with optimization of single-valued functions. Some good examples of this type of optimization is the well known maximization of profit, maximization of consumer's utility or minimization of costs. Multi-objective optimization deals with optimization of multi-valued (vector) functions. A good example of this

* Ph.D. Candidate, Institute of Economic Studies, Charles University in Prague.

type of optimization is welfare economics with determining the maximal utility over many consumers, while each has his or her own utility or the comparison of many medical treatments while there are many parameters that are measured and we have to combine them somehow to determine the superior treatment. Of course the extent to which optimization methods are described is very restricted as the scope of the book is relatively broad and its aim is surely not to explain the details of the background mathematics. The scope of optimization theory is very large and it would be a big mistake to get an impression, that almost everything dealing with optimization has already been solved. A typical example of very complicated optimization problem, that generally cannot be solved fast enough is the famous traveling salesman problem. In fact, mathematics is able to solve easily and elegantly only problems with several (sometimes not very real) assumptions (usually differentiability and concavity of the maximized functions). The function to be optimized is usually called an objective function.

Optimization theory can be also divided in another (and very important) way on unconstrained and constrained optimization. Unconstrained optimization is generally easier and deals with the maximization or minimization of functions on the whole set, in which they are defined. Constrained optimization is on the other hand given by optimizing functions on some part of the set, in which they are defined. This part is usually defined by a set of other functions, equalities or inequalities. A typical inequality that holds in economics is the non-negativeness of the variables.

Classical optimization theory is divided into linear programming (objective function is linear and the set of feasible solutions is a simplex) and nonlinear programming (any maximization or minimization of a function on some set, which is not a linear programming problem). The nonlinear programming can be divided in two parts: convex programming and non-convex programming. Convex programming deals with the minimization of convex functions or the maximization of concave functions on convex sets (technically, linear programming is a subset of convex programming). Problems of convex programming are generally easier to solve as a convex, differentiable function has at most one general minimum on a convex set. In addition to convex programming, there are also quasi-convex programming (the function to be minimized is non-convex and the set of feasible solutions is convex), integer programming (any function to be optimized on a finite or countable infinite set of distinct integer points) and other special optimization problems. There are specific problems of nonlinear optimization, such as problems of quadratic programming (convex programming with quadratic function to be optimized), separable programming (the set of feasible solutions is given by separable functions), geometric programming (the set of feasible solutions is given by functions in the form of polynomials with positive coefficients—so called posynomials) or fractional programming (the set of feasible solutions is given by functions in the form of fraction of two functions).

The theory is presented in a more or less mathematical manner, with definitions, theorems and their proofs. However, it is not as rigorous and mainly not as self-contained as purely mathematical books dealing with optimization techniques. The author does not provide all the relevant information, but always references where it can be found. I consider this approach sufficient (and probably also more convenient) for a book intended mainly for economists.

The book is not intended as a study of the computational aspects of solution techniques, however the theory is briefly described, starting with the introduction of the Lagrange function and continuing with the generalization by Kuhn and Tucker. Sometimes, when the proof is too long or not descriptive enough, it is omitted and only referenced. The Kuhn-Tucker conditions require at least all the functions defining the set of feasible solutions and the optimized function to be continuously differentiable on the set of feasible solutions. However, an additional assumption must be fulfilled for the Kuhn-Tucker conditions to be necessary. These conditions are usually referred to as the regularity conditions, and there are many of them. The author describes just one, the Slater condition and in addition references literature. It seems to be sufficient for a book intended mainly for applications in economic theory as the regularity conditions are in most economic applications automatically fulfilled. From the point of view of optimization theory itself, the book can be thought of as an introduction, and does not include some more recent developments, such as the introduction of invex functions in relation to the sufficiency of the Kuhn-Tucker conditions.

The economic applications of the Kuhn-Tucker conditions mentioned in the book are the following:

1. *Peak load pricing* deals with the problem of pricing of a commodity within a day in order to minimize the costs while the demand for this commodity changes throughout the day.

2. *Revenue maximization under a profit constraint* deals with maximization of a firm's total revenue subject to a profit constraint assuming advertising expenditure, which has an impact to the revenues.

3. *Behavior of the firm under regulatory constraint* deals with monopoly-profit maximization under a regulatory constraint of the maximal "fair rate of return" imposed by government.

4. *Environmental regulation: Standard as a set level of emissions* deals with maximizing profit under pollution standards in the form of maximal level of emissions permitted, while the level of emissions depends on the level of output (i.e. through the production function on the levels of inputs) and on the level of abatement expenditure (expenditure for development of more environment friendly technologies).

5. *Environmental regulation: Standard as emissions per unit of output* deals with maximizing profit under pollution standards in the form of maximal level of emissions per unit of output permitted, while the level of emissions depends on the level of output (i.e. through the production function on the levels of inputs) and on the level of abatement expenditure (expenditure for development of more environment friendly technologies).

6. *Environmental regulation: Standard as emissions per unit of a specified input* deals with maximizing profit under pollution standards in the form of maximal level of emissions per unit of one of the inputs permitted, while the level of emissions depends on the level of output (i.e. through the production func-

tion on the levels of inputs) and on the level of abatement expenditure (expenditure for development of more environment friendly technologies).

Convex programming is a specific part of the general nonlinear optimization theory, however it is usually treated separately as the Kuhn-Tucker conditions become also sufficient for the minimization problem. Convex programming deals with minimizing convex functions on a convex set of feasible solutions. To solve the minimization problem of convex programming the solution provided by the Kuhn-Tucker conditions is sufficient and necessary when at least one of the regularity constraints is met. Hence the first part of this chapter is devoted to basic theorems of mathematical analysis dealing with convexity and concavity of continuously differentiable functions (Hessian matrix and its semi-definiteness or indefiniteness is studied).

The Slater condition for convex programming problem is mentioned and explicitly expressed and Slater theorem is presented as a more general version of the Kuhn-Tucker theorem (no differentiability of functions is needed).

The author defines quasiconvex functions and shows that the Kuhn-Tucker conditions are sufficient conditions solving the convex programming problem and can be generalized to sufficient conditions for more general quasiconvex functions.

The author introduces the duality theory in convex programming using an example from the foundations of economic theory—the allocation of scarce resources among alternative uses (as the primal problem) vs. pricing (as the corresponding dual problem). A specific part of convex programming is quadratic programming, where some specific solution techniques can be used for finding the solution (Wolfe algorithm, Lemke algorithm), which are not studied in this book.

Linear programming is given as the minimization or maximization of a linear function over a set of feasible solutions given by linear constraints, so that the set is convex (in fact the set is a convex polyhedron). The optimal solution turns out to be either unique, or there can be an infinite number of solutions, or there can be no solution at all. Linear programming is the simplest part of convex programming, although not always computationally easily solved. The Kuhn-Tucker conditions for linear programming problems are necessary and sufficient, the regularity constraint is fulfilled. However, the Kuhn-Tucker conditions are relatively difficult when dealing with thousands or millions of variables. Many differentiations would have to be performed, while the solution is always in the frontier of the set of feasible solutions and moreover it is always in a corner (extreme) point if it exists. Therefore special computational techniques ranging from the general method of simplex developed by G. Dantzig to special algorithms of flows in networks were used.

The simplex algorithm is described, theoretically justified and illustrated on an example of a farmer's production problem (including graphical solution). A separate part is devoted to duality in liner programming, including an economic interpretation of the dual problem and dual variables. As the Kuhn-Tucker conditions are both sufficient and necessary for the general problem of linear programming, the author shows the relationship between the dual constraints and the Kuhn-Tucker conditions.

One interesting property arising in some linear programming economic models is illustrated in a section entitled "The More-for-Less Paradox". This paradox is very interesting as minimizing the total cost of producing certain amounts of outputs, given a matrix of technologies and fixed prices of inputs can result in higher value of the cost function, than if producing higher amounts of outputs. It is explained, that this property of the linear programming holds if and only if every optimal solution of the dual problem has at least one negative dual variable. The proof of this theorem was independently derived in two articles, one of which is by Chobot and Turnovec.

The important sections are again those with economic applications. There are three applications analyzed in detail:

1. *The theory of comparative advantage* deals with maximization of world output under constraints that the production factors in each country can not exceed the amount disposable and under the non negativity constraints for amount of outputs.

2. *The Giffen paradox* deals with the well known paradox of Giffen illustrated on the famous example by A. Marshall. It deals with a traveler using either a boat or train to travel a distance, while he has a given budget he can spend. Traveling by train costs some fixed amount per kilometer and traveling by boat as well. By increasing the price of traveling by boat, ceteris paribus, the part traveled by boat within the optimal solution increases.

3. *The Leontief pollution model* deals with the famous Leontief model applied to pollution produced by different industries. Some further analysis of the model via introduction of effluent taxes is performed.

Data envelopment analysis (DEA) is a relatively new field of optimization theory focused on efficiency and productivity analysis. It is not new in terms of mathematical programming and it does not escape the theory and solution techniques of mathematical programming described in the previous sections of the book, but it is innovative in terms of economic analysis. It is rather an application of mathematical programming, than a distinct part of the optimization theory.

The basic idea is based on a nonparametric approach for frontier estimation in the sense that it does not require any specific functional form. In DEA, every deviation from the production frontier is treated as inefficiency and provides a single measure of efficiency even when dealing with multiple inputs and outputs. A very important fact is that DEA obviates the need to assign prespecified weights to the inputs and outputs to be able to measure the efficiency. The basic idea is the introduction of so called decision-making units (DMUs), which can be on the production frontier (are efficient) or below (are inefficient). Each DMU not on the frontier is scaled against a linear or convex combination of the DMUs on the production frontier faced closest to it. For each inefficient DMU, DEA identifies the sources and level of inefficiency for each of the inputs and outputs.

I consider DEA to be one of the most important recent applications of mathematical programming. The difference between technical and allocative efficiency is explained. While technical efficiency stands for the production situation, when the units are on the production frontier using the best attainable production technology, allocative efficiency also takes into account the prices of all the inputs. Therefore, there are technically efficient production strategies which are not allocatively efficient, while the opposite is not true. The overall efficiency stands for the product of the technical efficiency and allocative efficiency.

DEA treats the observed inputs and outputs as given constants and chooses values of the input and output weights for a particular DMU such that the efficiency (ratio of the weighted sum of outputs and weighed sum of inputs) is maximized subject to the less-than-unity constraints. The indicated maximization provides the most favorable weighting for each DMU that the constraints allow. There are two basic types of DEA models: constant returns to scale (CRS) and variable returns to scale (VRS). Both these models lead into problems of mathematical (fractional) programming and can be efficiently solved by transformation to linear programming model using dual problem.

The author devotes one part of this chapter to measuring the "ecoefficiency", where undesirable outputs are introduced (such as various kinds of pollution). An example of DEA analysis of the ecoefficiency of the industries in 16 OECD countries is presented.

Geometric programming is a specific part of nonlinear mathematical programming, where the objective function and the constraint functions are posynomials (polynomials with positive coefficients). This type of problem was widely studied. The basic difference between the solution technique used for the geometric programming problem and the technique used for other mathematical programming problems is in the direction of solution. Instead of searching first for the optimal values of the variables in the objective function, we search for the optimal value of the objective function and then determine the resulting solution to the variables.

The general geometric programming problem can be transformed via exponential transformation of the primal variables to a convex programming problem. By logarithm transformation of the resulting functions a programming problem is obtained and its dual is transformed to a problem that can be solved relatively easily.

The second part of the book is devoted to the multi-objective optimization. The author starts with the economic applications, to introduce the reader to the subject. There are six applications briefly mentioned:

1. *Welfare economics* dealing with the total economic welfare of a society, which depends on the welfare level of all of its individuals. The total utility is given as a vector of the individual utilities and the basic concept of the maximization of this vector is the so-called Pareto optimality, i.e. a situation is Pareto optimal if no individual utility can be increased, without decreasing the utility of at least one other individual. The author defines an equilibrium point and welfare maximum point and social welfare function (SWF) and explains the difference between the utilitarian and Rawlsian SWF and shows the multi-objective optimization of both the approaches (utilitarian and Rawlsian) simultaneously.

2. *Quantitative economic policy* dealing with general policy making and introduction of the interactive multi-objective programming.

3. *Optimal monetary policy* dealing with the optimal monetary policy taking into account two strategic variables: inflation

and output (hitting inflation and output gap). The model introduced is based on the new Keynesian model with a forward looking Phillips curve and a forward looking IS (investment-saving) curve. The central bank would minimize the expected violation of inflation hitting (computed based on the squares of deviations of real inflation form the target) and the expected output gap.

4. *Optimal behavior of a monopolist facing a bi-criteria objective function* dealing with a monopolist maximizing the profit and revenues at the same time.

5. *Leontief pollution model with multiple objectives* deals with the Leontief model with two objectives: minimization of pollution and maximization of revenue.

6. *A nonlinear model of environmental control* deals with a nonlinear model of environmental damage costs, which are expected to rise progressively with an increase in pollution. The model maximizes consumption and at the same time minimizes environmental damage costs.

The Kuhn-Tucker conditions for the multi-objective programming problem are introduced to find the Pareto-Koopmans efficient solution and some further theory is briefly studied, including the duality theory of multi-objective programming. Namely, the parametric optimization is shown to be interconnected to the multi-objective optimization. Besides the Pareto-Koopmans efficiency (or optimality), lexicographical minimization is mentioned. Besides the general introduction to the multi-objective optimization, the author deals separately with the multi-objective linear programming and the multi-objective geometric programming.

I consider the content of the book highly interesting, mainly data envelopment analysis and multi-objective optimization. There are thousands of articles dealing with multi-objective optimization and the general problem is still very current. Data envelopment analysis is very applicable in practical problems, where the efficiency of distinct decision making units have to be measured and compared.

The book is a well written introduction to optimization theory and contains well organized and exceptionally interesting examples of economic applications, including further theory motivated by the mentioned applications. The book also provides relevant references to literature including books and articles related to each studied topic. It is clear that the book has been written by a fully competent author with good experience in teaching. On the other hand, I would not recommend the book to students of mathematics, who are interested mainly in the theory itself, as the book is clearly intended for students of economic theory. This book should not give students of economic theory the impression that there is nothing more to optimization theory than what is presented. I very much appreciate the inclusion of recent issues, but on the other hand I would welcome at least a brief introduction to combinatorial optimization, stochastic optimization and integer programming.