

## A Flow Approach to Bankruptcy Problems

Rodica Branzei\*, Giulio Ferrari\*\*, Vito Fragnelli†, Stef Tijs‡

Received 9 July 2008; Accepted 25 August 2008

**Abstract** In this note we represent a classical bankruptcy problem as a standard flow problem on a simple network and implement some known division rules from the bankruptcy literature via suitable cost functions in the related minimum cost flow problem.

**Keywords** Allocation rules, bankruptcy problems, flow problems

**JEL classification** G33

### 1. Introduction

A classical bankruptcy problem arises from a situation where some agents have claims on the available estate to be divided among them such that each agent may receive a nonnegative amount that cannot exceed his or her claim. Several division rules have been introduced to solve the classical bankruptcy problem and some extensions or generalizations of it. There is a huge literature on bankruptcy problems and related division rules. Here we mention the papers by O’Neill (1982), Aumann and Maschler (1985), Curiel, Maschler and Tijs (1987), Young (1987, 1994, 1998), Kaminski (2000), Borm, Hamers and Hendrickx (2001), Herrero and Villar (2001) and Thomson (2003). The area of applications of bankruptcy problems is impressively large, including different real life problems for which a bankruptcy-like approach has been proved beneficial.

In this note we represent the classical bankruptcy problem as a standard flow problem where each feasible monetary flow corresponds then to a possible solution of the bankruptcy problem.

A standard flow problem arises from a flow situation which is modeled as a network with two special nodes, the source and the sink, and on whose arcs there are capacity restrictions. In the case of a standard flow problem the main interest is in a maximal flow through the network from source to sink. A min cost flow problem arises from a standard flow situation where on each of the arcs there is also a cost function besides the capacity restrictions and each node has a demand or a supply. In a min cost flow

---

\* Alexandru Ioan Cuza University, Faculty of Computer Science, Carol I Bd. 11, 700483 Iași, Romania. Phone +40232201529, E-mail: branzeir@infoiasi.ro.

\*\* University of Genoa, Department of Mathematics, Via Dodecaneso 35, 16146 Genoa, Italy. Phone +390103536941, E-mail: ferrari@dima.unige.it.

† University of Eastern Piedmont, Department of Advanced Sciences and Technologies, Via V.Bellini 25/G, 15100 Alessandria, Italy. Phone +390131360224, E-mail: vito.fragnelli@mfn.unipmn.it.

‡ Tilburg University, Center and Department of Econometrics and Operations Research, P.O. Box 90153, 5000 Tilburg, the Netherlands. University of Genoa, Department of Mathematics, Via Dodecaneso 35, 16146 Genoa, Italy. Phone +31134662348, E-mail: S.H.Tijs@uvt.nl.

problem the main interest is a cheapest flow through the network, satisfying both the capacity constraints on the arcs and the demand or supply of the nodes. To get insight into the question how a min cost flow problem can be solved we refer the reader to Ahuja et al. (1993).

By defining a suitable min cost flow problem to represent a bankruptcy problem we can drive the solution of a standard flow problem towards a particular solution of the related bankruptcy problem. We provide different cost functions associated to the minimum cost flow problem such that the solutions coincide with the most well known solutions for bankruptcy problems.

The outline of this note is as follows. In Section 2 we introduce the representation of a classical bankruptcy problem as a standard flow problem and implement some division rules from the bankruptcy literature via related min cost flow problems. We conclude in Section 3.

## 2. Bankruptcy rules via min cost flow problems

A classical (one-claim) bankruptcy problem arises from a situation where an estate  $E$  has to be divided among several claimants, each of them with a claim on the estate, and the total claim exceeds the available estate. The set  $N$  of claimants is of the form  $\{1, 2, \dots, n\}$ . Each claimant  $i \in N$  advances one claim  $c_i$  on the estate  $E$ . A bankruptcy problem is an ordered triplet  $(N, E, c)$  where  $E \in \mathbb{R}_+$ ,  $c \in \mathbb{R}_+^n$  and  $E \leq c_1 + c_2 + \dots + c_n$ .

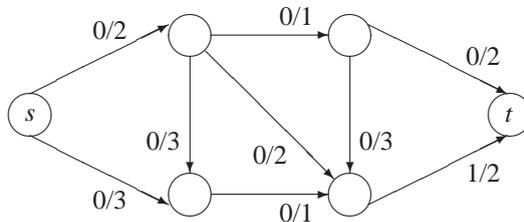
A solution of a bankruptcy problem is a vector  $x = (x_1, x_2, \dots, x_n)$  such that:

$$0 \leq x_i \leq c_i, \quad \text{for each } i \in N,$$

$$\sum_{i=1}^n x_i = E,$$

where  $x_i$  can be interpreted as the part of the estate  $E$  assigned to claimant  $i$ . In this section we represent a classical bankruptcy problem as a standard flow problem.

**Figure 1.** Standard flow problem

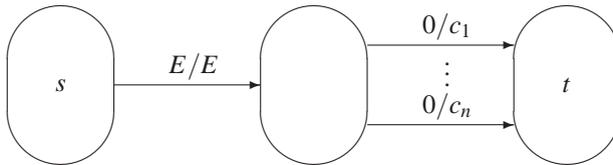


In a standard flow problem there is a network with two special nodes, the source and the sink, the first with no entering arcs and the latter without outgoing arcs. The arcs have minimal and maximal capacity constraints. A flow is a function that assigns a nonnegative real value to each arc, respecting the capacity constraints and such that for

each node different from the source and the sink the sum of the values for the entering arcs equals the sum of the values for the outgoing arcs. Figure 1 shows a graphical representation of a standard flow problem, where  $s$  and  $t$  are respectively the source and the sink and the notation  $a/b$  on the arcs stands for *min capacity/max capacity*.

A classical bankruptcy problem can be easily represented as a standard flow problem by constructing a network like in Figure 2.

**Figure 2.** Classical bankruptcy problem



Here the source of the monetary flow corresponds to the bankrupt agent or company and the sink corresponds to the group of claimants, each one being associated to one of the arcs entering the sink. An immediate advantage of such a representation is that each feasible monetary flow corresponds to a solution of the bankruptcy problem  $(N, E, c)$ . The simplicity of the resulting network problem allows us to manage also complex and sophisticated solutions.

Further, we relate a pair consisting of a classical bankruptcy problem  $(N, E, c)$  and a division rule  $f$  to a min cost flow problem.

A division rule is a function  $f$  which assigns to any bankruptcy problem  $(N, E, c)$  a vector  $f(N, E, c) \in \mathbb{R}^n$  that is a solution of the bankruptcy problem. Well known division rules which we use in this note are: the proportional rule (*PROP*), the constrained equal award rule (*CEA*), the constrained equal loss rule (*CEL*), the Talmudic rule (*TAL*) and the adjusted proportional rule (*APROP*).

We start by briefly describing these rules:

- (i) The  $i$ -th coordinate of  $PROP(N, E, c)$  is given by

$$PROP_i(N, E, c) = \left( \sum_{j \in N} c_j \right)^{-1} c_i E, \quad i \in N.$$

- (ii) The  $i$ -th coordinate of  $CEA(N, E, c)$  is given by

$$CEA_i(N, E, c) = \min\{c_i, \alpha\}, \quad i \in N,$$

where  $\alpha$  is the unique real number such that  $\sum_{i \in N} CEA_i(N, E, c) = E$ .

- (iii) The  $i$ -th coordinate of  $CEL(N, E, c)$  is given by

$$CEL_i(N, E, c) = \max\{c_i - \beta, 0\}, \quad i \in N,$$

where  $\beta$  is the unique real number such that  $\sum_{i \in N} CEL_i(N, E, c) = E$ .

(iv) The  $i$ -th coordinate of  $TAL(N, E, c)$  is given by

$$TAL_i(N, E, c) = \begin{cases} CEA_i(N, E, \frac{c}{2}) & \text{if } E < \frac{1}{2} \sum_{j \in N} c_j \\ c_i - CEA_i(N, E', \frac{c}{2}) & \text{if } E \geq \frac{1}{2} \sum_{j \in N} c_j \end{cases}, i \in N,$$

where  $E' = \sum_{j \in N} c_j - E$ .

(v) The  $i$ -th coordinate of  $APROP(N, E, c)$  is given by

$$APROP_i(N, E, c) = m_i(N, E, c) + PROP_i(N, E', c'), i \in N,$$

where  $m_i(N, E, c) = \max\{0, E - \sum_{j \in N \setminus \{i\}} c_j\}$  is the minimal right of claimant  $i \in N$ ,  $E' = E - \sum_{j \in N} m_j$  and  $c'_i = \min\{E', c_i - m_i(N, E, c)\}$ ,  $i \in N$ .

The main aim of this note is to show that various division rules from the bankruptcy literature can be “implemented” via suitable cost functions of the corresponding min cost flow problem. Moreover, different priorities or rights of the claimants may be tackled using suitable cost functions on the arcs.

We start with a general description of a natural procedure for obtaining the cost functions of a min cost flow problem such that the generated solution coincides with the solution obtained by using a specific division rule from the bankruptcy literature. Our procedure is based on the fact that it is possible, by assigning suitable cost functions  $k_1, k_2, \dots, k_n$  on the arcs, to drive the flow in a min cost flow problem towards solutions arising from specific bankruptcy rules. The inspiration source has been the hydraulic model of Kaminski (2000), where the potential induced by the gravity force allows water to suitably fill the vessels whose form and configuration depend on the specific division rule. In our min cost flow situation the task of dividing the estate according to a given bankruptcy rule is fulfilled by endowing the arcs of the network with the right cost functions.

Given claims  $c_1, \dots, c_n$ , for each  $E \in [0, \sum_{i \in N} c_i]$  let a bankruptcy rule assign  $r_1(E), \dots, r_n(E)$  to the claimants. Then a Lagrangian analysis implies that we have to take the cost functions  $k_1, k_2, \dots, k_n$  such that

$$k'_i(r_i(E)) = k'_n(r_n(E))$$

for each  $i \in N$  and each  $E \in [0, \sum_{i \in N} c_i]$ .

Let  $s = r_n(E)$ ; then  $r_i(E) = r_i(r_n^{-1}(s)) = p_i(s)$  and, consequently,  $k'_i(p_i(s)) = k'_n(s)$ ,  $i = 1, \dots, n - 1$ . Now, assuming that  $p'_i(s) \neq 0$  for each  $i \in N$ , we obtain for each  $t$  the relation

$$\int_0^t k'_n(s) ds = \int_0^t k'_i(p_i(s)) ds = \int_0^t \frac{k'_i(p_i(s))p'_i(s)}{p'_i(s)} ds = \int_0^{p_i(t)} \frac{k'_i(u)}{p'_i(p_i^{-1}(u))} du,$$

where  $u = p_i(s)$ . Note that these integrals relate the various cost functions.

Based on the above described procedure we have Theorem 1.

**Theorem 1.** *Let  $(N, E, c)$  be a classical bankruptcy problem. The division rules PROP, CEA, CEL, TAL and APROP can be implemented via a min cost flow problem.*

**Proof.** We only give for each rule a set of suitable cost functions which have been obtained by applying the procedure described in this section.

(i) PROP:  $k_i(x_i) = \frac{x_i^2}{c_i}, i \in N.$

(ii) CEA:  $k_i(x_i) = x_i^2, i \in N.$

(iii) CEL:  $k_i(x_i) = (c_i - x_i)^2, i \in N.$

(iv) For the Talmudic rule we can distinguish among the two cases  $E < \frac{1}{2} \sum_{i \in N} c_i$  and  $E \geq \frac{1}{2} \sum_{i \in N} c_i$ . In the first case we have

$$TAL: k_i(x_i) = x_i^2, i \in N,$$

setting the minimal capacity to 0 and the maximal capacity of the arcs corresponding to the claimants to  $\frac{1}{2}c_i$  instead of  $c_i, i \in N$ .

In the second case we have

$$TAL: k_i(x_i) = (c_i - x_i)^2, i \in N,$$

setting the minimal capacity of the arcs corresponding to the claimants to  $\frac{1}{2}c_i$  instead of 0 and the maximal capacity to  $c_i, i \in N$ .

(v) APROP:  $k_i(x_i) = \begin{cases} 0 & \text{if } x_i \leq m_i \\ \frac{(x_i - m_i)^2}{c'_i} & \text{if } x_i > m_i \end{cases}, i \in N. \quad \square$

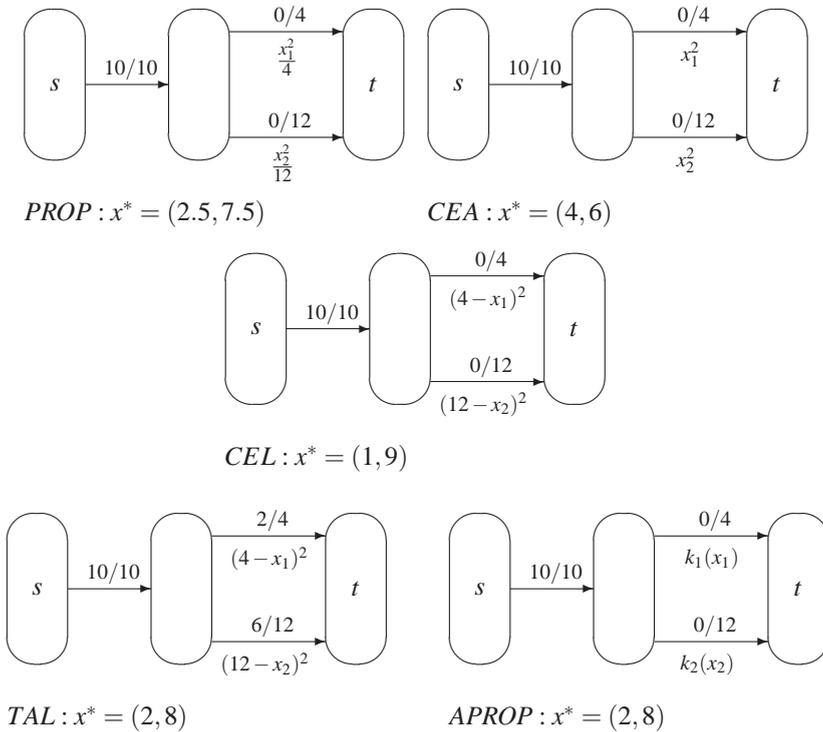
**Example 1.** *Consider the bankruptcy problem with  $N = \{1, 2\}; E = 10; c = (4, 12)$ . The flow problems associated to the above five rules are depicted in Figure 3 (the notations for the arcs are “min capacity/max capacity” above the arc and “cost function” below the arc). The cost functions used for the rule APROP are:*

$$k_1(x_1) = \begin{cases} 0 & \text{if } x_1 \leq 0 \\ \frac{x_1^2}{4} & \text{if } x_1 > 0 \end{cases}, \quad k_2(x_2) = \begin{cases} 0 & \text{if } x_2 \leq 6 \\ \frac{(x_2 - 6)^2}{4} & \text{if } x_2 > 6 \end{cases}$$

### 3. Concluding remarks

In this note we focus on the connection between (one-claim) bankruptcy problems and flow problems. Here we go more deeply into this interesting connection than we did previously (Branzei, Ferrari, Fragnelli and Tijs, 2004). The connection between a classical bankruptcy problem and a flow problem can be extended by considering more complicated settings from the bankruptcy literature. Multi-claim bankruptcy

Figure 3. Flow problems associated to the division rules



problems (cf. Kaminski, 2000) are particularly interesting because different seniorities of the claimants and the application of the absolute priority rule (or violations of this rule) play a key role in the flow representation of such problems leading to more sophisticated networks. Furthermore, the recent developments in the theory of interval bankruptcy rules (cf. Branzei and Alparslan Gök, 2008; Branzei and Dall’Aglio, 2008; Branzei, Dall’Aglio and Tijs, 2008) provide incentives for extending our connection between classical bankruptcy problems and flow problems to interval bankruptcy problems (cf. Branzei, Dimitrov and Tijs, 2003; Branzei and Alparslan Gök, 2008).

The representation of a bankruptcy problem as a standard flow problem can be also helpful for generating solutions for extensions and/or generalizations of the classical bankruptcy problems arising from various economic situations. In particular, the flow approach may be useful for managing more complex situations in which the bankrupt firm faces its creditors by taking into account its debtors. In these situations other arcs, each related to a debtor, may be added among the source and the second node.

We have shown that by a particular choice of the cost functions in a min cost flow problem one can obtain the solution corresponding to a specific well known rule from

the bankruptcy literature. Nevertheless, the metaphor of a min cost flow problem can be helpful to “generate” new rules for rationing problems as well and for managing situations in which the claimant agents have different priorities or particular rights on the existing estate. The first case may be managed using cost functions that assign lower costs for higher priorities while for the second case a suitable minimal capacity may be used for representing the rights of the agents.

**Acknowledgment** The authors gratefully acknowledge two anonymous referees and the Editors for their useful comments and suggestions.

## References

- Ahuja, R. K., Magnanti, T. L. and Orlin, J. B. (1993). *Network Flows*. New Jersey, Prentice Hall.
- Aumann, R. J. and Maschler, M. (1985). Game Theoretic Analysis of a Bankruptcy Problem from the Talmud. *Journal of Economic Theory*, 36, 195–213.
- Borm, P., Hamers, H. and Hendrickx, R. (2001). Operations Research Games: A Survey. *TOP*, 9, 139–216.
- Branzei, R. and Alparslan Gök, S. Z. (2008). Bankruptcy Problems with Interval Uncertainty. Ankara, Middle East Technical University, Institute of Applied Mathematics. (Preprint No. 111)
- Branzei, R. and Dall’Aglio, M. (2008). Allocation rules incorporating interval uncertainty. (mimeo)
- Branzei, R., Dall’Aglio, M. and Tijs, S. (2008). Interval game theoretic division rules. (mimeo)
- Branzei, R., Dimitrov, D., and Tijs, S. (2003). Shapley-like Values for Interval Bankruptcy Games. *Economics Bulletin*, 3, 1–8.
- Branzei, R., Ferrari, G., Fragnelli, V. and Tijs, S. (2004). Bankruptcy Problems and Minimum Cost Flow Problems. Genova, Università di Genova, Dipartimento di Matematica. (Preprint No. 518)
- Curiel, I. J., Maschler, M. and Tijs, S. (1987). Bankruptcy Games. *Zeitschrift für Operations Research*, 31, 143–159.
- Herrero, C. and Villar, A. (2001). The Three Musketeers: Four Classical Solutions to Bankruptcy Problems. *Mathematical Social Sciences*, 42, 307–328.
- Kaminski, M. M. (2000). ‘Hydraulic’ Rationing. *Mathematical Social Sciences*, 40, 131–155.
- O’Neill, B. (1982). A Problem of Rights Arbitration from the Talmud. *Mathematical Social Sciences*, 2, 345–371.
- Thomson, W. (2003). Axiomatic and Game-theoretic Analysis of Bankruptcy and Taxation Problems: A Survey. *Mathematical Social Sciences*, 45, 249–297.

Young, H. P. (1987). On Dividing an Amount According to Individual Claims or Liabilities. *Mathematics of Operations Research*, 12, 398–414.

Young, H. P. (1994). Cost Allocation. In Aumann, R. J. and Hart, S. (eds.), *Handbook of Game Theory*, vol. II. Amsterdam, North Holland, 1193–1235.

Young, H. P. (1998). Distributive Justice in Taxation. *Journal of Economic Theory*, 48, 321–335.