Czech Economic Review 8 (2013) 84–101
Acta Universitatis Carolinae Oeconomica

Refining the Information Function Method: Instrument and Application

Dan Qin*

Received 3 March 2014; Accepted 18 August 2014

Abstract Information function method is a powerful tool for analyzing the information requirements of social welfare functions. However, the original information function provides only a coarse description of information structure. In this article, we propose a refinement of this method by changing the range of the information function. We also analyze the role of partially relevant information in preference aggregation through an application of the refined version of the information function method.

Keywords Preference aggregation, independence condition, information function, partially relevant information
JEL classification D71

1. Introduction

Arrow's impossibility theorem (Arrow 1963) is a landmark in the history of thought. This extremely robust theorem, like all theorems in social choice, studies possible ways of aggregating individual preferences into *social preference*. The theorem states that there exists no aggregating function (social welfare function) to construct social preference from individual preferences if this function is required to satisfy basic principles of (i) unanimity (Weak Pareto: if everyone strictly prefers x to y, so be society), (ii) democracy (Non-Dictatorship: no individual can always realize his strict preferences), and (iii) informational parsimony (Independence of Irrelevant Alternatives: social preferences on any subset of alternative space depend only on individual preferences over this subset).

A common means of escaping from the Arrow's impossibility theorem is to discard one of these requirements. A particularly active branch of this line is to drop or relax the Independence of Irrelevant Alternatives (IIA). It is well known that dropping IIA opens a floodgate of possibilities for well-behaved social welfare functions to exist. Borda's method, among many other positional methods, is an example. Relaxing IIA, rather than dropping it completely, attracts less attention in the literature. Among the exceptions, Campbell and Kelly (2000) provide a seminal contribution to this line. They propose *information function* and *relevance mapping* to study the information requirements of social welfare functions.

* Waseda University, Graduate School of Economics, 1-104 Totsukamachi, Shinjuku-ku, Tokyo, 169-8050, Japan. Phone: 81-80-3128-3636, E-mail: Lucifier1031@foxmail.com.

The information function is a form of mapping associated with each social welfare function, whose domain is the set of all unordered pairs of distinct alternatives and whose range is the set of all collections of subsets of alternative space. Given a social welfare function, its associated information function maps each pair of distinct alternatives to a collection of subsets S of the set of alternatives such that individual preferences restricted to any S in this collection contain enough information to determine social order over this pair. Campbell and Kelly (2000) show that this collection satisfies the intersection property, i.e., if S_1 and S_2 belong to this collection, then their intersection $S_1 \cap S_2$ also belongs to this collection. In light of this property, a minimal element (minimal in terms of set inclusion) exists in every collection. The minimal element, referred to as *relevant set*, is the smallest set containing enough information to rank the pair. *Relevance mapping* is then defined as a function from each pair of distinct alternatives to its relevant set. For example, for any social welfare function satisfying IIA, the relevant set of any $\{x,y\}$ is $\{x,y\}$. Positional rules like Global Borda's rule are often considered as an opposite extreme in terms of information parsimony. For Global Borda's rule, the relevant set of any $\{x,y\}$ is the set of all alternatives.

This instrument has been proven to be a powerful tool in the analysis of information structure (see Campbell and Kelly 2007a, 2007b, 2009). However, the information requirement of a social welfare function and its associated relevance mapping are not as tightly connected as they seem to be. Remember that individual preferences are binary relations, i.e. subsets of $X \times X$. Relevance mapping, on the other hand, describes a subset S of the set of alternatives such that the restriction of individual's preferences, i.e. $R_i \cap S \times S$, is enough to determine the social ranking of this pair of alternatives. Therefore, there are gaps between the relevance mapping and the genuine informational requirement. Consider Global Borda's rule: ranking $\{x,y\}$ does not require information on $\{w,z\}$ when $\{x,y\} \cap \{w,z\} = \emptyset$. The relevant set associated with $\{x,y\}$, however, is the entire alternative space. One cannot tell if individual preferences on $\{w,z\}$ are really needed to rank $\{x,y\}$ from its associated relevant set. In this sense, the original information function and relevance mapping only provide a coarse description of the amount of information used. The reason behind is that not all restrictions on preference information can be expressed in the form of $R|S \times S$ for $S \subseteq X$. For example, there does not exist $S \subseteq X$ such that $S \times S = \{(x,y),(y,x),(x.z),(z,x)\}$. To address this issue, we propose a refinement of the information function method. The refinement is made by changing the range of the information function. Our refined information function maps each $\{x,y\}$ to a collection of subsets of the set of all unordered paris of distinct alternatives. We show that this refinement is genuine, i.e., it can represent more structure than the original information function and relevance mapping.

Despite the theoretical interest, we show that the refinement also has instrumental value: it facilities the analysis of certain kinds of preference information. Hansson (1973) discusses a special kind of preference information: partially relevant information. For a pair of distinct alternatives $\{x,y\}$, preference information over $\{\{x,z\},\{y,z\} : z \notin \{x,y\}\}$ is called partially relevant. Clearly, the original version of information function lacks the ability to analyze this kind of information. Literatures in economic environment also mainly discuss information on pairs of alternatives (See Fleurbaey, Suzumura, and Tadenuma 2005a, 2005b).[1] This work is one attempt to fill this blank in Arrovian framework. By applying the refined version of relevance mapping, we consider the information structure of social welfare functions satisfying certain axiomatic properties. In particular, we analyze the role of partially relevant information (Hansson 1973) in constructing neutral or anonymous social welfare functions.

The remainder of this article is organized as follows. Section 2 introduces notation and definitions. Section 3 proposes our refinement of the information function method. Basic structure of relevance mapping is also discussed here. Section 4 analyzes the role of partially relevant information by applying our refined information function and relevance mapping. Section 5 presents concluding comments.

2. Framework

We begin with the set X of alternatives and the set $N = \{1, 2, \ldots, n\}$ of individuals. Throughout this note, X and N are assumed to be finite. $|X|$ is assumed to be greater or equal to 3 and $|N|$ is assumed to be greater or equal to 2. A binary relation R on X is a subset of $X \times X$. Let P and I denote asymmetric and symmetric parts of R respectively. Individual preferences are represented by binary relations on X: xP_iy means individual i strictly prefers x to y. Throughout this article, individual preferences are assumed to be linear orders (complete, transitive, and antisymmetric), whereas social preference is assumed to be an ordering (complete and transitive).[2] Denote the set of all linear order (respectively, ordering) on X as $L(X)$ (respectively, $O(X)$).

A social welfare function f is a mapping from $L^n(X)$ to $O(X)$, which aggregates individual preferences into the social preference. Elements in $L^n(X)$ are called profiles, denoted by p. Denote $f(p)$ as social preference generated by f from p. Let $f^*(p)$ denote the asymmetric part of $f(p)$. When there is no ambiguity, we also use R and R' to represent $f(p)$ and $f(p')$, respectively.

For any $S \subseteq X$, denote the set of all unordered pairs of distinct alternatives in S as *Int*, i.e., $Int(S) = \{\{x,y\} : x,y \in S \ \& \ x \neq y\}$. In particular, let $\hat{X} = Int$ denote the set of all unordered pairs of distinct alternatives. For any $S \subseteq \hat{X}$, let $p|S$ and $R|S$ denote the restriction of p and R on S respectively. That is

$$R|S = R \cap \bigcup_{a \in S} a \times a,$$

and for $p = \{R_1, \ldots, R_n\}$,

$$p|S = \{R_1|S, \ldots, R_n|S\}.$$

For any set S, let $Pow(S)$ denote its power set.

---

[1] The informational basis of most allocation rules in economic enviroment is closely related to the indifference curve. Information on indifference curve passing through x is equivalent to information of upper and lower contour sets of x, which represents preference information on sets of pairs of alternatives rather than sets of alternatives.

[2] A binary relation R is complete if for any $x, y \in X$, either xRy or yRx; R is transitive if $xRy \ \& \ yRz \Rightarrow xRz$ for any $x, y, z \in X$; R is antisymmetric if $xIy \Rightarrow x = y$.

The next order of business is to introduce some properties on social welfare functions. A social welfare function f satisfies Weak Pareto if for any $\{x,y\} \in \hat{X}$ and any $p \in L^n(X)$, $[\forall i \in N, xP_iy]$ implies $xf^*(p)y$. A dictator is an individual i such that xP_iy implies $xf^*(p)y$ for any $\{x,y\} \in \hat{X}$ and any $p \in L^n(X)$. A social welfare function is Non-Dictatorial if there exists no dictator.

A social welfare function f satisfies Independence of Irrelevant Alternatives (IIA) if for any $\{x,y\} \in \hat{X}$ and any p, $p' \in L^n(X)$,

$$p|\{\{x,y\}\} = p'|\{\{x,y\}\} \Rightarrow f(p)|\{\{x,y\}\} = f(p')|\{\{x,y\}\}.$$

Arrow (1963) shows that no Non-Dictatorial social welfare function satisfies Weak Pareto and IIA.

Non-Dictatorial can be strengthened to require the symmetric treatment of individuals. Let $\sigma : N \leftrightarrow N$ denote a permutation of N. Each permutation of N induces a permutation on profiles as

$$\sigma(R_1,R_2,\ldots,R_n) = (R_{\sigma(1)},R_{\sigma(2)},\ldots,R_{\sigma(n)}).$$

A social welfare function f satisfies *anonymity* if for any permutation σ and any profile p, $f(\sigma(p)) = f(p)$.

Similarly, any permutation $\mu : X \leftrightarrow X$ of alternatives induces a permutation on individual preferences defined by $\mu(x)\mu(R)\mu(y) \iff xRy$. This in turn induces a permutation on profiles as

$$\mu(R_1,R_2,\ldots,R_n) = (\mu(R_1),\mu(R_2),\ldots,\mu(R_n)).$$

A social welfare function satisfies *neutrality* if for any profile p and any permutation μ on X, $f(\mu(p)) = \mu(f(p))$.

These are standard axiom in social choice theory; for detailed discussion of these axioms, see e.g. Taylor (2005).

3. Refining the information function method

Campbell and Kelly (2000) define the information function associated with a social welfare function f as a mapping from $\hat{X}$ to $Pow(Pow(X))$:

$$\Psi^f(\{x,y\}) = \{S \subseteq X : p|Int(S) = p'|Int(S) \Rightarrow f(p)|\{\{x,y\}\} = f(p')|\{\{x,y\}\}\}$$

Individual preferences over any element of $\Psi^f(\{x,y\})$ are sufficient to rank $\{x,y\}$. Campbell and Kelly (2000) show that $\Psi^f(\{x,y\})$ satisfies the intersection property, i.e., $S_1,S_2 \in \Psi^f(\{x,y\})$ implies $S_1 \cap S_2 \in \Psi^f(\{x,y\})$. Because of the intersection property and the finiteness of X, for each $\{x,y\}$ there exists S^* such that $\Psi^f(\{x,y\}) = \{S : S^* \subseteq S \subseteq X\}$. This set is called the relevance set of $\{x,y\}$. Define ψ^f to be the mapping from each $\{x,y\}$ to its relevant set. ψ^f is called the relevance mapping for f. ψ^f maps each distinct pair to a subset of X such that this subset is the minimal set (in terms of set inclusion) required to rank this pair. For example, for any social welfare function satisfying IIA, $\psi^f(\{x,y\}) = \{x,y\}$. However, the information requirement of a social welfare function and its associated ψ^f are not tightly connected. In other words, there are social welfare functions with different information requirements while their associated relevance mappings are identical. Consider the following examples with $X = \{x,y,z\}$ and $N = \{1,2,\dots,n\}$.

Example 1. Define social welfare function f_1 as follows. For any $p \in L^n(X)$,

(i) $f_1(p)|\{\{x,y\}\} = R_1|\{\{x,y\}\}$;

(ii) If zP_2x and zP_2y then $zf_1^*(p)x$ & $zf_1^*(p)y$; in other cases, $f_1(p)|\{\{x,z\},\{y,z\}\} = R_1|\{\{x,z\},\{y,z\}\}$.

Example 2. Define social welfare function f_2 as follows. For any $p \in L^n(X)$,

(i) $f_2(p)|\{\{x,y\}\} = R_1|\{\{x,y\}\}$;

(ii) If $[xP_1y \;\&\; zP_2x]$ or $[yP_1x \;\&\; zP_2y]$ then $zf_2^*(p)x$ & $zf_2^*(p)y$; in other cases, $f_2(p)|\{\{x,z\},\{y,z\}\} = R_2|\{\{x,z\},\{y,z\}\}$.

It is easily verifiable that

$$\begin{aligned} &\psi^{f_1}(\{x,y\}) = \psi^{f_2}(\{x,y\}) = \{x,y\} \\ \text{and} \quad &\psi^{f_1}(\{y,z\}) = \psi^{f_1}(\{x,z\}) = \psi^{f_2}(\{y,z\}) = \psi^{f_2}(\{x,z\}) = \{x,y,z\}. \end{aligned}$$

Hence, $\psi^{f_1} = \psi^{f_2}$.

Observe, however, f_1 does not need preference information over $\{\{x,y\}\}$ to rank $\{y,z\}$ and $\{x,z\}$. Two profiles differing only on $\{\{x,y\}\}$ must generate identical ranking on $\{y,z\}$ and $\{x,z\}$ by f_1. On the other hand, f_2 requires preference information over $\{\{x,y\}\}$ to rank $\{y,z\}$ and $\{x,z\}$. To see this, consider two profiles

$$\begin{aligned} p: &\quad xP_1yP_1z \;\&\; yP_2zP_2x \\ p': &\quad yP_1xP_1z \;\&\; yP_2zP_2x \end{aligned}$$

Observe that p and p' only differ over $\{\{x,y\}\}$ while we have $zf_2^*(p)y$ and $yf_2^*(p')z$.

These examples illustrate that ψ only provides a coarse description of the information requirement of social welfare functions. This is because the range of the information function is $Pow(Pow(X))$, which lacks the ability to differentiate between sets like $\{\{x,y\},\{x,z\}\}$ and $\{\{x,y\},\{y,z\}\}$. For each pair of distinct alternatives, ψ describes a collection of self-dependent subset of X such that restriction of individual preferences on ψ is enough to rank the pair. Because individual and social preferences are subsets of $X \times X$, the restriction of R on S normally defined as $R \cap S \times S$, which is equivalent to $R|Int(S)$ in our definition.[3] Therefore, relevance mapping can also be interpreted as follows: individual preferences over $Int(\psi(\{x,y\}))$ are sufficient to rank $\{x,y\}$. However, there might be elements of $Pow(\hat{X}) \setminus \{Int(S) : S \subseteq X\}$ possessing the

[3] $R|Int(S) = R \cap \bigcup_{a \in Int} a \times a = R \cap S \times S$

same property. To rectify this deficiency, we refine the information function method by changing its range to $Pow(Pow(\hat{X}))$:

$$\hat{\Psi}^f(\{x,y\}) = \{S \subseteq \hat{X} : p|S = p'|S \Rightarrow f(p)|\{\{x,y\}\} = f(p')|\{\{x,y\}\}\}$$

Similar to the original information function, this version also satisfies a number of properties. Firstly, for any $S_1 \subseteq S_2 \subseteq \hat{X}$, $S_1 \in \hat{\Psi}^f(\{x,y\})$ implies $S_2 \in \hat{\Psi}^f(\{x,y\})$. This is obvious by definition. Secondly, although it is less transparent, this version of the information function also satisfies the intersection property. All proofs can be found in the Appendix.

Theorem 1. *For any social welfare function f and its associated information function $\hat{\Psi}$, for any $\{x,y\} \in \hat{X}$ and any $S_1,S_2 \subseteq \hat{X}$, $S_1,S_2 \in \hat{\Psi}^f(\{x,y\})$ implies $S_1 \cap S_2 \in \hat{\Psi}^f(\{x,y\})$.*

Remark 1. *Theorem 1 is stronger than the original intersection property. To see this, note that $S_1,S_2 \in \Psi^f(\{x,y\})$ is equivalent to $Int(S_1), Int(S_2) \in \hat{\Psi}^f(\{x,y\})$ by definition. By theorem 1, $Int(S_1) \cap Int(S_2) \in \hat{\Psi}^f(\{x,y\})$. Observe that $Int(S_1) \cap Int(S_2) = Int(S_1 \cap S_2)$. Therefore, $Int(S_1 \cap S_2) \in \hat{\Psi}^f(\{x,y\})$, which is equivalent to $S_1 \cap S_2 \in \Psi^f(\{x,y\})$.*

Reversely, consider two subsets of $\hat{X}$:

$$S_1 = \{\{x,y\},\{x,z\}\}; S_2 = \{\{x,y\},\{y,z\}\}$$

By the original intersection property, one cannot get $S_1 \cap S_2 \in \hat{\Psi}^f(\{x,y\})$ from $S_1,S_2 \in \hat{\Psi}^f(\{x,y\})$. In this sense, theorem 1 strengthens the original intersection property.

Remark 2. *The intersection property of the original information function does not require any domain condition, not even the normal assumption that individual preferences are transitive. Our intersection property is achieved by explicitly using transitivity of individual preferences as a premise.*

Because of the intersection property and the finiteness of $\hat{X}$, for each $\{x,y\}$ there exists a minimal (minimal in terms of set inclusion) set S^* such that $\hat{\Psi}^f(\{x,y\}) = \{S : S^* \subseteq S \subseteq \hat{X}\}$, referred to as the relevant set of $\{x,y\}$. Define $\hat{\psi}^f$ to be the mapping from each $\{x,y\}$ to its relevant set. $\hat{\psi}^f$ describes the minimal information requirement of its associated social welfare function. The next theorem show the relation between $\hat{\psi}^f$ and ψ^f.

Theorem 2.

(i) *For any social welfare functions f, f' and any $\{x,y\} \in \hat{X}$, $\hat{\psi}^f(\{x,y\}) = \hat{\psi}^{f'}(\{x,y\})$ implies $\psi^f(\{x,y\}) = \psi^{f'}(\{x,y\})$.*

(ii) *There are social welfare functions with identical ψ whereas their associated $\hat{\psi}$ are different for some $\{x,y\}$.*

Remark 3. *Theorem 2 illustrates that $\hat{\psi}^f$ is a genuine refinement of ψ^f. Part (i) of Theorem 2 shows that ψ^f cannot describe more structure of information requirement than $\hat{\psi}^f$. Part (ii) of Theorem 2 shows that $\hat{\psi}^f$ provides more detailed description of the information structure than ψ^f in some cases.*

Remark 4. *One implication worth noting here is that* $\{w,z\} \in \hat{\psi}^f(\{x,y\})$ *if and only if there exist two profiles* p *and* p' *such that* $p|\hat{X} \setminus \{\{w,z\}\} = p'|\hat{X} \setminus \{\{w,z\}\}$ *and* $f(p)|\{\{x,y\}\} \neq f(p')|\{x,y\}$ *(and of course* $p|\{w,z\} \neq p'|\{\{w,z\}\}$*).*[4] *Further, we can find two profiles such that they only differ in one individual's preference on* $\{w,z\}$ *and social orders generated from these two profiles differ on* $\{x,y\}$*. To see this, simply note that because the only difference of* p *and* p' *is on* $\{w,z\}$*, individuals having different preferences on* $\{w,z\}$ *in* p *and* p' *must rank* w *and* z *adjacently in both profiles. Let us call this group of individuals* C*. Construct new profile* p^1 *from* p' *by switching the position of* $\{w,z\}$ *for one individual in* C*. If the social order on* $\{x,y\}$ *changes then* p *and* p^1 *is what we want to find. Otherwise construct* p^2 *from* p^1 *by switching* $\{w,z\}$ *for another individual in* C*. Because* N *is finite, we can eventually find two profiles* p^i *and* p^{i+1} *with the desired property.*

The next order of business is to discuss the structure of relevance mapping when its associated social welfare function satisfies Arrovian properties. The implication of IIA is obvious: if a social welfare function satisfies IIA then $\hat{\psi}_i^f(\{x,y\}) \subseteq \{\{x,y\}\}$ for any $\{x,y\} \in \hat{X}$.

If Weak Pareto is imposed, then $\{x,y\} \in \hat{\psi}^f(\{x,y\})$ for any $\{x,y\} \in \hat{X}$. To see this, assume the contrary that $\{x,y\} \notin \hat{\psi}^f(\{x,y\})$. Consider profile p in which every individual put x at the top and y just below x. By Weak Pareto, xPy. Construct p' from p by transposing the position of x and y. Since $\{x,y\} \notin \hat{\psi}^f(\{x,y\})$, we have $xP'y$, which violates Weak Pareto. If a Paretian social welfare function f satisfies IIA, then $\hat{\psi}_i^f(\{x,y\}) = \{\{x,y\}\}$ for any $\{x,y\} \in \hat{X}$.

It is also worthwhile to note that Pareto and transitivity together force further structure on $\hat{\psi}^f$ as stated in the next theorem, which will serve as an auxiliary result in the next section.

Theorem 3. *For any Paretian social welfare function* f*:*

(i) *For any* $x,y,z \in X$*, if* $\hat{\psi}^f(\{x,y\}) \setminus \{x,y\} \not\subseteq \hat{\psi}^f(\{y,z\})$ *and there exists* $S \subset X$ *such that*

- (a) $z \in S$, $x,y \in X \setminus S$,
- (b) *for any* $u \in S$ *and* $v \in X \setminus S$, $\{u,v\} \notin \hat{\psi}^f(\{x,y\})$,

then $\{x,z\} \in \hat{\psi}^f(\{y,z\})$.

(ii) *For any* $x,y,z,w \in X$*, if* $\hat{\psi}^f(\{x,y\}) \setminus \{\{x,y\}\} \not\subseteq \hat{\psi}^f(\{w,z\})$ *and there exist* $S_1, S_2 \subset X$ *such that*

- (a) $w \in S_1$, $z \in S_2$, $x,y \in X \setminus (S_1 \cup S_2)$, *and* $S_1 \cap S_2 = \emptyset$,
- (b) *for any* $u \in S_1$ *and* $v \in X \setminus S_1$, $\{u,v\} \notin \hat{\psi}^f(\{x,y\})$,
- (c) *for any* $u \in S_2$ *and* $v \in X \setminus S_2$, $\{u,v\} \notin \hat{\psi}^f(\{x,y\})$,

then $\{w,x\} \notin \hat{\psi}^f(\{w,z\})$ *implies* $\{z,y\} \in \hat{\psi}^f(\{w,z\})$.

[4] See also part one of Lemma 1 in the Appendix.

4. Application: partially relevant information

Positional methods like Global Borda's rule are favourably contrasted with pairwise methods because they generate transitive social preferences. Global Borda's rule satisfies all Arrovian conditions except IIA. Further, it also satisfies stronger conditions like Neutrality and Anonymity. Its success is commonly attributed to the fact that it uses much more preference information than social welfare functions satisfying IIA. In this section, we show that it is a special kind of preference information that is indispensable for a social welfare function to satisfy Neutrality or Anonymity. This particular kind of information is called *partially relevant* information, as first discussed by Hansson (1973, p. 41):

> The weak independence conditions to be studied in this section are based on the assumption that there are degrees of relevance for preferences between different alternatives. If we adopt Arrow's interpretation of A as the set of all conceivable alternatives and B as the set of all available alternatives, it seems safe to say that preferences between two elements of B are relevant and preferences between two elements of A-B irrelevant for the choice from the set B. But what about preference between one element in B and one in A-B? Arrow chooses to treat them as irrelevant too, but admits on p.19 of [2] that they may be of some use, as e.g. in determining strength of preferences. We will now introduce a notation to deal with this distinction and then see what happens if we take these halfway relevant preferences into account.

Following Hansson (1973), we call preference information on pairs like $\{x,z\}$ *partially relevant* information and preference information on pairs like $\{w,z\}$ *fully irrelevant* information when the social ranking of $\{x,y\}$ is being discussed. It is easily verifiable that Global Borda's rule uses all partially relevant information and does not use any fully irrelevant information. To examine the role of partially relevant information, we define some independence conditions.

Definition 1. *A social welfare function f satisfies Independence* if for any $\{x,y\} \in \hat{X}$, $\hat{\psi}^f(\{x,y\}) \subseteq Int(\{x,y\}) \cup Int(X \setminus \{x,y\})$.*

Independence* excludes all partially relevant information in aggregating process. We also define some weaker versions.

Definition 2. *A social welfare function f satisfies Weak Independence* if for any $x,y,z \in X$, $\{x,z\} \in \hat{\psi}^f(\{x,y\})$ implies that $\forall w \in X \setminus \{x,y,z\}, \{x,w\}, \{y,w\} \notin \hat{\psi}^f(\{x,y\})$.*

Weak Independence* says that one can use a small part of partially relevant information.

Definition 3. *A social welfare function f satisfies Minimal Independence* if there exist $x,y,z \in X$ such that $\{x,z\} \notin \hat{\psi}^f(\{x,y\})$.*

Minimal Independence* says that the social welfare function must be independent of at least some partially relevant information. We also define an independence condition which rejects the use of fully irrelevant information.

Definition 4. *A social welfare function f satisfies Hansson Independence if for any $x,y \in X$, $Int(X \setminus \{x,y\}) \not\subseteq \hat{\psi}^f(\{x,y\})$.*

Observe that positional rules satisfy Hansson Independence, i.e., no fully irrelevant information is used in the decision process. Meanwhile, even the weakest conditions in its line, Minimal Independence*, is violated by all positional methods. In other words, every bit of partially relevant information is required. It is tempting to ask if there exist social welfare functions possessing similar axiomatic advantages as positional methods, i.e., satisfying Neutrality or Anonymity, while using less partially relevant information. The next theorem examines the possibility for neutral rules.

Theorem 4. *If a social welfare function satisfies Weak Pareto, Neutrality, and Minimal Independence*, then it is dictatorial.*

Theorem 4 shows that all partially relevant information is indispensable if we want a Paretian Non-dictatorial social welfare function to satisfy Neutrality. We now turn to Anonymity.

Theorem 5. *No Paretian social welfare function satisfies Anonymity and Independence*.*

For three alternatives, involving a small part of partially relevant information allows a social welfare function to be anonymous. The next example illustrates this claim. Let xP_Ny mean x being unanimously preferred to y.

Example 3. For $X = \{x,y,z\}$, let social order be determined as follows:

(i) If yP_Nx then yPx; otherwise xPy.

(ii) If yP_Nz then yPz; otherwise zPy.

(iii) If $\neg yP_Nx$ then $zP_Nx \Rightarrow zPx$ and $\neg zP_Nx \Rightarrow xPz$.

(iv) If yP_Nx then $xP_Nz \Rightarrow xPz$ and $\neg xP_Nz \Rightarrow zPx$.

It is easily verifiable that this social welfare function is well defined and satisfies Weak Pareto, Anonymity, and Hansson Independence (trivially). Its associated relevance mapping is as follows:

$$\hat{\psi}^{f_1}(\{x,y\}) = \{\{x,y\}\}$$
$$\hat{\psi}^{f_1}(\{y,z\}) = \{\{y,z\}\}$$
$$\hat{\psi}^{f_1}(\{x,z\}) = \{\{x,y\},\{x,z\}\}$$

This example shows that if $|X| = 3$ then a small amount of partially relevant information is enough to bring us from dictatorship all the way to an anonymous social welfare function.[5] However, for $|X| \geq 4$, even a bit more partially relevant information is not enough if fully irrelevant information is precluded.

Theorem 6. *For $|X| \geq 4$, no Paretian social welfare function satisfies Hansson Independence, Weak Independence***, and Anonymity.*

Unlike Neutrality, Relaxing Weak Independence* to Minimal Independence* is enough for an anonymous social welfare function to exist.

Example 4. For $X = \{x, y, z, w\}$, let social order be determined in the following way:

(i) For $u \in \{y, z, w\}$, xPu if $|\{i : xP_iu\}| \geq |\{i : uP_ix\}|$ and uPx otherwise.

(ii) For $u, v \in \{y, z, w\}$, if uPx & xPv by step 1 then let transitivity dictate.

(iii) If there exists $u \in \{z, w\}$ such that uPx & yPx or xPu & xPy by step 1 and 2 then order y and u in the following way: yPu if $|\{i : yP_iu\}| \geq |\{i : uP_iy\}|$ and uPy otherwise.

(iv) If transitivity of step 1, 2, and 3 cannot determine social order on w, z then zPw if $|\{i : zP_iw\}| \geq |\{i : wP_iz\}|$ and wPz otherwise.

By construction, this social welfare function is well defined and satisfies Weak Pareto and Anonymity. Minimal Independence* is also verifiable by observing that $\hat{\psi}^f(\{x, u\}) = \{\{x, u\}\}$ for any $u \in X \setminus \{x\}$.[6]

5. Concluding remarks

The objective of this article is twofold. Firstly, we refined the information function method to describe a more detailed information structure of social welfare functions. The major purpose of constructing information function and relevance mapping is to provide tools to analyze the information requirement of social welfare functions. Therefore, information function and relevance mapping should describe as much informational structure as possible. However, the original version of the information function and relevance mapping provides only a coarse description of the information requirement of social welfare functions. The range of the original information function, $Pow(Pow(X))$, makes it impossible to differentiate between certain kinds of information requirement. For example, if a social welfare function uses individual preferences over $\{\{x, y\}, \{y, z\}\}$ to rank $\{x, y\}$ then the original relevance set of $\{x, y\}$ is $\{x, y, z\}$. One cannot tell from this relevant set if individual preferences over $\{x, z\}$ are required to rank $\{x, y\}$. Recall that examples 1 and 2 are associated with identical information function and relevance mapping while the actual informational structure is different. To address this problem, we propose a refinement of the information function by changing its range to $Pow(Pow(\hat{X}))$, the set of all collections of subsets of the set of

[5] This social welfare function also satisfies Hansson Independence because Hansson Independence is vacuum when $|X| = 3$.

[6] This social welfare function violates Hansson Independence because $\hat{\psi}^f(w, z) = \hat{X}$.

unordered distinct pairs of alternatives. This refinement enables the information function and relevance mapping to differentiate between relevance sets like $\{\{x,y\},\{x,z\}\}$ and $\{\{x,y\},\{y,z\}\}$.

Our refinement is interesting from both the theoretical and the applicational point of view. Theoretically, we showed that it is a genuine refinement of the original information function and relevance mapping method: social welfare functions associated with identical refined relevance mapping must be associated with an identical original version of relevance mapping; reversely, there are social welfare functions associated with an identical original relevance mapping while their refined relevance mappings are different. Besides, the intersection property itself describes a significant amount of the structure of the information requirement and our intersection property is essentially stronger than the original version. This strengthening is achieved by making explicit use of the assumption that *individual* preferences are transitive.

The refinement is not only interesting from a theoretical point of view. The new version of the information function and relevance mapping can facilitate the analysis of certain types of information. For example, extensive analysis of Hansson's (1973) partially relevant information can now be done by applying our information function method. This leads to the second objective of this article.

We examined the role of partially relevant information in constructing a well-behaved social welfare function by applying the refined version of the information function and relevance mapping. The role of partially relevant information is worth exploring because almost all known neutral and anonymous social welfare functions, e.g., positional methods like Global Borda's rule, use all partially relevant information and are independent of fully irrelevant information. This information structure requires an explanation. We show that the entire set of partially relevant information is indispensable in constructing neutral social welfare functions. For anonymous social welfare functions, the requirement also goes well beyond a marginal amount of partially relevant information when the number of alternatives is at least 4. In economic environment, preference information on pairs of alternatives have been studied intensely. Our study on partially relevant information is one attempt to fill the blank in Arrovian framework.

There are limits to our work which should be borne in mind here. The informational structure of social welfare functions is a complex matter and our analysis is far from being exhaustive. We have focused on what seems to us the most natural way of describing the information requirement, namely, the preference information on subsets of $\hat{X}$. There may be other ways to describe the informational structure of social welfare functions. For example, Powers (2005) considers positional information, which describes information requirement on certain positions of individual preferences. It would be interesting to combine these approaches. One can also consider local information, i.e., allowing the use of information on the set of alternatives which are ranked adjacently to x,y. The purpose of this article would be well served if it could open a gate to these enticing avenues.

Acknowledgment The author would like to express particular thanks to Yasumi Matsumoto, Kohei Kamaga, and two anonymous referees for their valuable advices. The author is also in debt to Anthony Newell for editing the English of this article.

References

Arrow, K. (1963). *Social Choice and Individual Values*, 2nd edition. New Haven, Connecticut, Yale University Press.

Campbell, D. and Kelly, J. (2000). Information and Preference Aggregation. *Social Choice and Welfare*, 17(1), 3–24.

Campbell, D. and Kelly, J. (2007a). Social Welfare Functions that Satisfy Pareto, Anonymity, and Neutrality, but not Independence of Irrelevant Alternatives. *Social Choice and Welfare*, 29(1), 69–82.

Campbell, D. and Kelly, J. (2007b). Pareto, Anonymity, and Independence: Four Alternatives. *Social Choice and Welfare*, 29(1), 83–104.

Fleurbaey, M., Suzumura, K. and Tadenuma, K. (2005a). Arrovian Aggregation in Economic Environments: How Much Should We Know about Indifference Surfaces? *Journal of Economic Theory*, 124(1), 22–44.

Fleurbaey, M., Suzumura, K. and Tadenuma, K. (2005b). The Informational Basis of the Theory of Fair Allocation. *Social Choice and Welfare*, 24(2), 311–341.

Hansson, B. (1973). The Independence Condition in the Theory of Social Choice. *Theory and Decision*, 4(1), 25–49.

Powers, R. (2005). Positional Information and Preference Aggregation. *Social Choice and Welfare*, 24(3), 575–583.

Taylor, A. D. (2005). *Social Choice and the Mathematics of Manipulation*. Cambridge, Cambridge University Press.

Appendix

Proof of Theorem 1.

(i) We first prove that $S_1, \hat{X} \setminus \{\{w,z\}\} \in \hat{\Psi}^f(\{x,y\}) \Rightarrow S_1 \setminus \{\{w,z\}\} \in \hat{\Psi}^f(\{x,y\})$. The case when $\{w,z\} \notin S_1$ is obvious. We consider the opposite case.

For any $p|S_1 \setminus \{\{w,z\}\} = p'|S_1 \setminus \{\{w,z\}\}$, we want to show $f(p)|\{\{x,y\}\} = f(p')|\{\{x,y\}\}$. The case $p|\{\{w,z\}\} = p'|\{\{w,z\}\}$ is trivial since $S_1 \in \hat{\Psi}^f(\{x,y\})$. Assume the contrary and divide N into four groups:

$$N_1 = \{i \in N : \exists\{v_1^i, v_2^i, \dots, v_s^i\}, wP_iz \;\&\; zP_i'v_1^iP_i' \dots P_i'v_s^iP_i'w\}$$
$$N_2 = \{i \in N : \exists\{u_1^i, u_2^i, \dots, u_t^i\}, zP_iw \;\&\; wP_i'u_1^iP_i' \dots P_i'u_t^iP_i'z\}$$
$$N_3 = \{i \in N : R_i|\{\{w,z\}\} = R_i'|\{\{w,z\}\}\}$$
$$N_4 = N \setminus (N_1 \cup N_2 \cup N_3)$$

N_4 is the set of individuals whose preferences of $\{w,z\}$ differ in p and p' while w,z is ranked adjacently in p'.

For $i \in N_1$, we say that $v \in \{v^i_1, v^i_2, \dots, v^i_s\}$ is connected with w if either $\{v,w\} \in S_1$ or there exists a sub-sequence $\{v^i_{\sigma(1)}, \dots, v^i_{\sigma(r)}\}$ of $\{v^i_1, v^i_2, \dots, v^i_s\}$ such that

$$\{v, v^i_{\sigma(1)}\}, \{v^i_{\sigma(1)}, v^i_{\sigma(2)}\}, \dots, \{v^i_{\sigma(r-1)}, v^i_{\sigma(r)}\}, \{v^i_{\sigma(r)}, w\} \in S_1.$$

We say that $v \in \{v^i_1, v^i_2, \dots, v^i_s\}$ is connected with z if either $\{z,v\} \in S_1$ or there exists a sub-sequence $\{v^i_{\sigma(1)}, \dots, v^i_{\sigma(r)}\}$ of $\{v^i_1, v^i_2, \dots, v^i_s\}$ such that

$$\{z, v^i_{\sigma(1)}\}, \{v^i_{\sigma(1)}, v^i_{\sigma(2)}\}, \dots, \{v^i_{\sigma(r-1)}, v^i_{\sigma(r)}\}, \{v^i_{\sigma(r)}, v\} \in S_1.$$

Similarly, for $i \in N_2$, we say that $u \in \{u^i_1, u^i_2, \dots, u^i_t\}$ is connected with z if either $\{u,z\} \in S_1$ or there exists a sub-sequence $\{u^i_{\sigma(1)}, \dots, u^i_{\sigma(q)}\}$ of $\{u^i_1, u^i_2, \dots, u^i_t\}$ such that

$$\{u, u^i_{\sigma(1)}\}, \{u^i_{\sigma(1)}, u^i_{\sigma(2)}\}, \dots, \{u^i_{\sigma(q-1)}, v^i_{\sigma(q)}\}, \{v^i_{\sigma(q)}, z\} \in S_1.$$

We say that $u \in \{u^i_1, u^i_2, \dots, u^i_t\}$ is connected with w if either $\{w,u\} \in S_1$ or there exists a sub-sequence $\{u^i_{\sigma(1)}, \dots, u^i_{\sigma(q)}\}$ of $\{u^i_1, u^i_2, \dots, u^i_t\}$ such that

$$\{w, u^i_{\sigma(1)}\}, \{u^i_{\sigma(1)}, u^i_{\sigma(2)}\}, \dots, \{u^i_{\sigma(q-1)}, v^i_{\sigma(q)}\}, \{v^i_{\sigma(q)}, u\} \in S_1.$$

Observe that no alternative can be connected with both w and z, otherwise the transitivity of individual preferences in p will be violated since $p|S_1 \setminus \{\{w,z\}\} = p'|S_1 \setminus \{\{w,z\}\}$.

Construct profile p^* from p' in the following way:

(a) For every $i \in N_1$, move alternatives connected with w just above z while keep the ranking inside this group unchanged. Then move alternatives connected with z just below w while keep the ranking inside this group unchanged. Lastly, move alternatives connected with neither w nor z to the top of the preferences while also keeping the ranking inside.

(b) For every $i \in N_2$, move alternatives connected with z just above w while keep the ranking inside this group unchanged. Then move alternatives connected with w just below z while keep the ranking inside this group unchanged. Lastly, move alternatives connected with neither w nor z to the top of the preferences while also keeping the ranking inside.

(c) For $i \in N_3 \cup N_4$, let $R^*_i = R'_i$.

By construction, $p^*|S_1 = p'|S_1$. Since $S_1 \in \hat{\Psi}^f(\{x,y\})$, we have $f(p')|\{\{x,y\}\} = f(p^*)|\{\{x,y\}\}$.

Observe that for $i \in N \setminus N_3$, w and z are ranked adjacently in p^*. Construct another profile p^{**} from p^* by switching w and z for $i \in N \setminus N_3$. Since $\hat{X} \setminus$

$\{\{w,z\}\} \in \hat{\Psi}^f(\{x,y\})$, we have $f(p^*)|\{\{x,y\}\} = f(p^{**})|\{\{x,y\}\}$.

By construction, we have $p^{**}|S_1 \setminus \{\{w,z\}\} = p^*|S_1 \setminus \{\{w,z\}\} = p'|S_1 \setminus \{\{w,z\}\} = p|S_1 \setminus \{\{w,z\}\}$ and $p^{**}|\{\{w,z\}\} = p|\{\{w,z\}\}$, which gives $p^{**}|S_1 = p|S_1$. Since $S_1 \in \hat{\Psi}^f(\{x,y\})$, we have $f(p^{**})|\{\{x,y\}\} = f(p)|\{\{x,y\}\}$. Therefore, $f(p')|\{\{x,y\}\} = f(p^*)|\{\{x,y\}\} = f(p^{**})|\{\{x,y\}\} = f(p)|\{\{x,y\}\}$.

(ii) Observe that $S_2 \in \Psi^f(\{x,y\})$ implies that $\hat{X} \setminus \{\{w,z\}\} \in \Psi^f(\{x,y\})$ for any $\{w,z\} \notin S_2$. By part 1, for any $\{w,z\} \in S_1 \setminus S_2$, $S_1 \setminus \{\{w,z\}\} \in \Psi^f(\{x,y\})$. Repeating the argument then gives $S_1 \setminus \{\{w,z\},\{u,v\}\} \in \Psi^f(\{x,y\})$ for any $\{u,v\} \in S_1 \setminus (S_2 \cup \{\{w,z\}\})$. Since X is finite, repeating this argument for certain times leads to $S_1 \setminus (S_1 \setminus S_2) \in \Psi^f(\{x,y\})$, which is equivalent to $S_1 \cap S_2 \in \Psi^f(\{x,y\})$. □

Proof of Theorem 2 depends on the following lemma.

Lemma 7. *For any social welfare function f,*

(i) (a) *For any $\{x,y\},\{z,w\} \in \hat{X}$, $\{z,w\} \in \hat{\psi}^f(\{x,y\}) \Leftrightarrow \hat{X} \setminus \{z,w\} \notin \hat{\Psi}^f(\{x,y\})$;*

(b) *For any $x,y,z \in X$, $z \in \psi^f(\{x,y\}) \Leftrightarrow X \setminus \{z\} \notin \Psi^f(\{x,y\})$;*

(ii) *For any $x,y,z \in X$, $z \notin \psi^f(\{x,y\})$ if and only if $\forall w \in X \setminus \{z\}, \{z,w\} \notin \hat{\psi}^f(\{x,y\})$;*

(iii) *For any $\{x,y\} \in \hat{X}$, $\hat{\psi}^f(\{x,y\}) \subseteq Int(\psi^f(\{x,y\}))$.*

Proof.

(i) We only prove part (a). Proof of part (b) is essentially the same. $\Rightarrow$: Assume the contrary that $\{z,w\} \in \hat{\psi}^f(\{x,y\})$ and $\hat{X} \setminus \{\{z,w\}\} \in \hat{\Psi}^f(\{x,y\})$. By theorem 1, $\hat{\psi}^f(\{x,y\}) \setminus \{\{z,w\}\} \in \hat{\Psi}^f(\{x,y\})$, which violates the minimality of $\hat{\psi}^f(\{x,y\})$.

$\Leftarrow$: This is equivalent to show that $\{z,w\} \notin \hat{\psi}^f(\{x,y\})$ implies $\hat{X} \setminus \{\{z,w\}\} \in \hat{\Psi}^f(\{x,y\})$. From $\{z,w\} \notin \hat{\psi}^f(\{x,y\})$ we can get $\hat{\psi}^f(\{x,y\}) \subseteq \hat{X} \setminus \{\{z,w\}\}$. Therefore, $\hat{X} \setminus \{\{z,w\}\} \in \hat{\Psi}^f(\{x,y\})$.

(ii) We show it by the following sequence of equivalences:

$$\begin{aligned}
z \notin \psi^f(\{x,y\}) &\Leftrightarrow X \setminus \{z\} \in \Psi^f(\{x,y\}) && \text{by part (i)}\\
&\Leftrightarrow Int(X \setminus \{z\}) \in \hat{\Psi}^f(\{x,y\}) && \text{by definition}\\
&\Leftrightarrow \hat{X} \setminus \bigcup_{w \in X \setminus \{z\}} \{\{w,z\}\} \in \hat{\Psi}^f(\{x,y\}) &&\\
&\Leftrightarrow \forall w \in X \setminus \{z\}, \hat{X} \setminus \{\{w,z\}\} \in \hat{\Psi}^f(\{x,y\}) &&\\
&\Leftrightarrow \forall w \in X \setminus \{z\}, \{w,z\} \notin \hat{\psi}^f(\{x,y\}) && \text{by part (i)}
\end{aligned}$$

(iii) By definition, $\psi^f(\{x,y\}) \in \Psi^f(\{x,y\})$ implies $Int(\psi^f(\{x,y\})) \in \hat{\Psi}^f(\{x,y\})$. By minimality of $\hat{\psi}^f(\{x,y\})$, $\hat{\psi}^f(\{x,y\}) \subseteq Int(\psi^f(\{x,y\}))$. □

Proof of Theorem 2.

(i) Assume $\hat{\psi}^f(\{x,y\}) = \hat{\psi}^{f'}(\{x,y\})$. It suffices to show that for any $z \in X$, $z \notin \psi^f(\{x,y\}) \Rightarrow z \notin \psi^{f'}(\{x,y\})$.

Assume that $z \notin \psi^f(\{x,y\})$. By (iii) of Lemma 1, $\hat{\psi}^f(\{x,y\}) \subseteq Int(\psi^f(\{x,y\}))$. Therefore, $\forall w \in X \setminus \{z\}, \{w,z\} \notin \hat{\psi}^f(\{x,y\})$. Because $\hat{\psi}^f(\{x,y\}) = \hat{\psi}^{f'}(\{x,y\})$, $\forall w \in X \setminus \{z\}, \{w,z\} \notin \hat{\psi}^{f'}(\{x,y\})$. By part (ii) of Lemma 1, $z \notin \psi^{f'}(x,y)$.

(ii) It suffices to provide an example. Consider f_1 and f_2 in Example 1 and 2 again. Recall that $\psi^{f_1} = \psi^{f_2}$. The relevance mapping $\hat{\psi}^{f_1}$ and $\hat{\psi}^{f_2}$, however, are as follows:

$$\hat{\psi}^{f_1}(\{x,y\}) = \hat{\psi}^{f_2}(\{x,y\}) = \{\{x,y\}\}$$

$$\hat{\psi}^{f_1}(\{x,z\}) = \hat{\psi}^{f_1}(\{y,z\}) = \{\{x,z\},\{y,z\}\}$$

$$\hat{\psi}^{f_2}(\{x,z\}) = \hat{\psi}^{f_2}(\{y,z\}) = \{\{x,z\},\{y,z\},\{x,y\}\}$$

□

Proof of Theorem 3. We only prove the first part. The second part is similar. Assume the contrary that $\{x,z\} \notin \hat{\psi}^f(\{y,z\})$. By Remark 4, $\hat{\psi}^f(\{x,y\}) \setminus \{\{x,y\}\} \not\subseteq \hat{\psi}^f(\{y,z\})$ implies that there exist two profiles p and p' differing only on some $\{s,t\} \notin \hat{\psi}^f(\{y,z\})$ while xPy and $yR'x$.

Construct p^* from p and p^{**} from p' in the following way:

(i) $p^*|S = p|S$; $p^*|X \setminus S = p|X \setminus S$;

(ii) $p^{**}|S = p'|S$; $p^{**}|X \setminus S = p'|X \setminus S$;

(iii) z is ranked just above x by every individual in p^*; z is ranked just below x by every individual in p^{**};

(iv) Let p^* and p^{**} agree on $\{\{u,v\} : u \in S \setminus \{z\} \;\&\; v \in X \setminus (S \cup \{x\})\}$.

The forth part is possible because s,t are both in S or both in $X \setminus S$ by assumption.

Observe that $p^*|\hat{\psi}^f(\{x,y\}) = p|\hat{\psi}^f(\{x,y\})$ and $p^{**}|\hat{\psi}^f(\{x,y\}) = p'|\hat{\psi}^f(\{x,y\})$. Therefore we have xP^*y and $yR^{**}x$. By Weak Pareto, zP^*x and $xP^{**}z$. By transitivity, zP^*y and $yP^{**}z$.

However, observe also that $p^*|\hat{X} \setminus \{\{s,t\},\{x,z\}\} = p^{**}|\hat{X} \setminus \{\{s,t\},\{x,z\}\}$. Therefore $f(p^*)|\{y,z\} = f(p^{**})|\{y,z\}$, which is a contradiction. □

To prove Theorem 4, we need an auxiliary lemma.

Lemma 8. *A neutral social welfare function satisfies Minimal Independence* if and only if it satisfies Independence*.*

Proof. It suffices to show that if there exist $x,y,z \in X$ such that $\{x,z\} \in \hat{\psi}^f(\{x,y\})$ then for any $w,u,v \in X$, $\{w,v\} \in \hat{\psi}^f(\{w,u\})$.

Recall that Remark 4 shows $\{x,z\} \in \hat{\psi}^f(\{x,y\})$ if and only if there exist p and p' differing only on $\{x,z\}$ while $f(p)|\{x,y\} \neq f(p')|\{x,y\}$. Assume, without loss of generality, xPy and $yR'x$.

Constructp^* from p by transposing x with w, y with u, and z with v. By Neutrality, wP^*u. Construct p^{**} from p' by transposing x with w, y with u, and z with v. By Neutrality, $uR^{**}w$. Observe that p^* and p^{**} only differ on $\{w,v\}$. By Remark 4, $\{w,v\} \in \hat{\psi}^f(\{w,u\})$. □

Proof of Theorem 4. By Lemma 2, Independence* is satisfied. Because Independence* is equivalent to IIA when $|X| \leq 3$, we consider cases when $|X| \geq 4$.

Assume the contrary that f is not dictatorial. By Arrow's theorem, IIA cannot hold. By Independence*, there exists $x,y,w,z \in X$ such that $\{w,z\} \in \hat{\psi}^f(\{x,y\})$. By Remark 4, there exist two profiles p and p' differ only on $\{w,z\}$ while xRy and $yP'x$. Because of Independence*, we can find such p and p' in which x,y are ranked above any other alternatives for every individual. Partition N into five groups according the preferences on $\{x,y\},\{w,z\}$. Profile p is then the following:

S_1	S_2	S_3	S_4	S_5
x	x	x	y	y
y	y	y	x	x
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
w	w	z	w	z
z	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	z	w	z	w
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Profile p' is the following:

S_1	S_2	S_3	S_4	S_5
x	x	x	y	y
y	y	y	x	x
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
z	w	z	w	z
w	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	z	w	z	w
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

The existence of this partition is guaranteed by Remark 4, which also shows that S_1 consists of only one individual.

Construct p^* from p' by transposing the position of x and w:

S_1	S_2	S_3	S_4	S_5
w	w	w	y	y
y	y	y	w	w
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
z	x	z	x	z
x	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	z	x	z	x
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

By Neutrality, yP^*w. Construct p^{**} from p^* in the following way:

S_1	S_2	S_3	S_4	S_5
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
w	w	z	y	z
z	x	$\vdots$	w	$\vdots$
x	y	w	x	y
y	$\vdots$	x	$\vdots$	w
$\vdots$	z	y	z	x
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

By Independence*, $yP^{**}w$. By Weak Pareto, $wP^{**}x$. By transitivity, $yP^{**}x$. However, this violates Independence* since $p^{**}|Int(\{x,y\}) \cup Int(X \setminus \{x,y\}) = p|Int(\{x,y\}) \cup Int(X \setminus \{x,y\})$. □

Proof of Theorem 5. For $|X| = 3$, Independence* is equivalent to IIA. By Arrow's theorem, Anonymity can not hold. For $|X| \geq 4$, pick $x, y, z \in X$ and construct $\hat{f} : L^n(\{x,y,z\}) \to O(\{x,y,z\})$ from f as follows. Pick a linear order l on $X \setminus \{x,y,z\}$. For any $\hat{p} \in L^n(\{x,y,z\})$, construct $\overline{p} \in L^n(X)$ in the following way:

(i) Let $\overline{p}|\{x,y,z\} = \hat{p}$;

(ii) Let $\overline{p}|X \setminus \{x,y,z\} = l$;

(iii) $\forall u \in \{x,y,z\}, v \in X \setminus \{x,y,z\}, \forall i \in X$, let $u\overline{P}_i v$.

Finally, let $\hat{f}(\hat{p}) = f(\overline{p})|\{x,y,z\}$. Observe that $\hat{f}$ inherits Weak Pareto and Anonymity from f. Further, if f satisfies Independence* then $\hat{f}$ satisfies IIA. By Arrow's theorem, $\hat{f}$ is dictatorial. Therefore, f cannot be anonymous. □

The proof of Theorem 6 depends on the following observation: In the presence of Hansson Independence and Weak Independence*, part (i) of Theorem 3 has a direct implication: $\{y,z\} \in \hat{\psi}^f(\{x,y\})$ implies that for any $w \in X \setminus \{x,y,z\}$

$$\{y,w\} \in \hat{\psi}^f(\{x,w\}) \text{ and } \{y,z\} \in \hat{\psi}^f(\{y,w\}) \vee \{x,w\} \in \hat{\psi}^f(\{y,w\}).$$

To see that $\{y,w\} \in \hat{\psi}^f(\{x,w\})$, consider $S = \{w\}$. By Hansson Independence, $\{y,z\} \notin \hat{\psi}^f(\{x,w\})$. By Weak Independence* and Hansson Independence, $\forall u \in S, v \in X \setminus S, \{u,b\} \notin \hat{\psi}^f(\{x,y\})$. Therefore, $\{y,w\} \in \hat{\psi}^f(\{x,w\})$ by part (i) of Theorem 3. The part $\{y,z\} \in \hat{\psi}^f(\{y,w\}) \vee \{x,w\} \in \hat{\psi}^f(\{y,w\})$ is similar.

Proof of Theorem 6. By Theorem 5, there must exist $x,y,z \in X$ such that $\{y,z\} \in \hat{\psi}^f(\{x,y\})$. By Weak Pareto, $\{x,y\} \in \hat{\psi}^f(\{x,y\})$ (Theorem 4 and Theorem 5). By Hansson Independence and Weak Independence*, $\hat{\psi}^f(\{x,y\}) = \{\{x,y\},\{y,z\}\}$. Since $|X| \geq 4$, there exists $w \in X \setminus \{x,y,z\}$. By Theorem 3, Hansson Independence, Weak Independence*, and $\hat{\psi}^f(\{x,y\}) = \{\{x,y\},\{y,z\}\}$ implies that

$$\{y,w\} \in \hat{\psi}^f(\{x,w\}) \text{ and } \{y,z\} \in \hat{\psi}^f(\{y,w\}) \vee \{x,w\} \in \hat{\psi}^f(\{y,w\}).$$

Case 1: If $\{y,z\} \in \hat{\psi}^f(\{y,w\})$ then $\hat{\psi}^f(\{y,w\}) = \{\{y,w\},\{y,z\}\}$ by Weak Pareto, Hansson Independence, and Weak Independence*. Again by Theorem 3, $\{x,y\} \in \hat{\psi}^f(\{x,w\})$, which violates Weak Independence* because $\{y,w\} \in \hat{\psi}^f(\{x,w\})$.

Case 2: If $\{x,w\} \in \hat{\psi}^f(\{y,w\})$ then $\hat{\psi}^f(\{y,w\}) = \{\{y,w\},\{x,w\}\}$. Theorem 3 then implies that $\{w,z\} \in \hat{\psi}^f(\{y,z\})$. Again, theorem 3 implies that $\{x,z\} \in \hat{\psi}^f(\{x,y\})$, violating Weak Independence* because we assume $\{y,z\} \in \hat{\psi}^f(\{x,y\})$. □