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On the Turnpike Property of the Modified Golden Rule

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Abstract Stationary and ergodic probability distribution is derived and the stochastic balanced path of capital accumulation is defined via corresponding martingale path. Moreover, stochastic version of asymptotic turnpike theorem is established in the sense of uniform topology and also neighborhood type of stochastic turnpike theorem is demonstrated in the sense of ergodic measure.

Keywords Stochastic growth, modified Golden Rule, turnpike theorem, ergodic consumption, martingale-path of capital accumulation
JEL classification C60, E13, E22

1. Introduction

As is well known, the Golden Rule or modified Golden Rule path of capital accumulation has attracted broad and also persistent research interest (see Cass 1972; Samuelson 1965; Ray 2009; Mitra and Ray 2012) after those seminal papers of Phelps (1961, 1962, 1965). The basic model of present study is based upon the framework of Mirman (1972), and the major goal of the paper is to extend existing studies under relatively weak conditions.

First, it is especially worth noting that we, to the best of our knowledge, have for the first time derived the stochastic balanced path of capital accumulation through martingale-path of capital accumulation when noting that martingale itself implies certain stability. Second, Theorem 2 proves that the martingale-path of capital accumulation will converge to the modified Golden Rule in the sense of uniform topology, thereby extending the classical asymptotic turnpike theorem to stochastic versions in a much stronger sense. Finally, Corollary 1 demonstrates a new type of turnpike theorem, which can be regarded as the generalized neighborhood turnpike theorem, i.e., the martingale path will spend almost all the time staying in any given neighborhood of the modified Golden Rule with a close-to-one probability.

The rest of the paper is organized as follows. Section 2 introduces the basic model. Section 3 is the major part of the current exploration and in there we have proved the turnpike theorems. Section 4 concludes with some remarks.

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2. The model

Throughout, we consider one-sector economic growth with uncertain technology. That is, we employ the production function $y_{t+1} = f(k_t, A_t)$ with k_t denoting the capital stock at time t and A_t representing independent and identically distributed random variable at time t, for $t = 0, 1, 2, \ldots$ Moreover, it is without loss of generality assumed that A_t is independent of k_t, and $A_t \geq 0$ for $t = 0, 1, 2, \ldots$ Let c_t denote consumption at time t, then we, in line with Mirman (1972), give the following stochastic difference equation that characterizes a discrete-time stochastic growth process,

$$k_t = f(k_{t-1}, A_{t-1}) - c_t \tag{1}$$

subject to $k_0 > 0$, $k_t \triangleq g(f(k_{t-1}, A_{t-1}))$ and $c_t \triangleq h(f(k_{t-1}, A_{t-1}))$, where g and h are continuously differentiable functions. And it is assumed that $k_{-1} \equiv k_0$, $A_{-1} \equiv A_0$, $c_{-1} \equiv c_0$ and $y_{-1} \equiv y_0$ throughout the paper.

Furthermore, as in Mirman (1972), we introduce the following assumptions with the purpose of making things much easier:

Assumption 1. *Suppose that* $f(0,A) = 0$, $\forall A > 0$, $f(k,0) = 0$, $\forall k \geq 0$, *and also* $f(k,\infty) = \infty$, $\forall k \in (0,\infty)$.

Assumption 2. *We assume that* $\frac{\partial f(k,A)}{\partial k} \triangleq f_1 > 0, \frac{\partial^2 f(k,A)}{\partial k^2} \triangleq f_{11} < 0, \frac{\partial f(k,A)}{\partial A} \triangleq f_2 > 0$, *and also* $f_1(0,A) = \infty, f_1(\infty,A) = 0$, $\forall A > 0$.

Assumption 3. *Here, and throughout the paper,* μ_t *is the Borel probability measure associated with the random variable* k_t, $\forall t = 0, 1, 2, \ldots$

Assumption 4. *Throughout, we consider almost surely bounded random shocks, i.e.,* $\mu_t(A_t = \infty) = 0$, $\forall t = 0, 1, 2, \ldots$

Under Assumption 1 to 3, it is easily seen that k_t is a Markov process which belongs to $\mathbb{R}_+ \triangleq [0,\infty]$. Moreover, we need:

Assumption 5. *Assume that for each consumption policy* $c = h(y) = \tilde{h}(k,A)$, *there exists a constant* $\gamma < 1$ *such that* $0 < h(y) < \gamma y$ *with* $y = f(k,A)$. *Additionally,* $\tilde{h}(0,A) = 0$, $\forall A > 0$, *and* $\tilde{h}(\cdot,\cdot)$ *is continuous in both arguments.*

Assumption 6. *Assume that the expected production function* $E_A[f(k,A)] = E_A[y] < \infty$ *for all* $k \in \mathbb{R}_+$, *and also* $\limsup_{k\to\infty} E_A\left[\frac{f(k,A)}{k}\right] = 0$.

Assumption 7. *Suppose that M, the set of finite signed measures defined on any subset of the compact set* $\mathbb{R}_+ \triangleq [0,\infty]$, *is equipped with the weak* topology. And* $M^+ \subseteq M$ *denotes the set of Borel probability measures, and* M^+ *is a weak* compact set with* $\mu_0 \in M^+$.

So, based on the above assumptions, Mirman (1972) asserts the following result:

Proposition 1. *For each consumption policy, there exists a stationary (or invariant) Borel probability measure* μ_∞, *having no mass at either zero or infinity.*

And, we can derive the following proposition:

Proposition 2. (Ergodic consumption) *If k_0 is initially distributed like the stationary (or invariant) Borel probability measure μ_∞, then we get:*

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}\tilde{h}(k_t,A_t)=\int_{\mathbb{R}_+}\tilde{h}(k,A)\mu_\infty(dk)\quad a.s.$$

That is to say, μ_∞ is also an ergodic Borel probability measure, and consumption policy $c_t \triangleq \tilde{h}(k_{t-1},A_{t-1})$ exhibits ergodic property for $\forall t=0,1,2,\ldots$

Proof. It follows from Assumption 4 to 6 that,

$$\int_{\mathbb{R}_+}\tilde{h}(k,A)\mu_\infty(dk)<\infty.$$

Then, an application of the Birkhoff Ergodic Theorem gives the desired result. □

3. Modified Golden Rule and turnpike theorems

The major goal of the present section is to derive the modified Golden Rule based upon the model introduced in Section 2 and analyze the corresponding turnpike properties of the modified Golden Rule. By (1), we get,

$$c_t=f(k_{t-1},A_{t-1})-k_t,$$

which, on the steady state, becomes,

$$c=f(k,A)-k.$$

Then, the corresponding FOC is,

$$\frac{\partial c}{\partial k}=f_1(k,A)-1=0. \tag{2}$$

So, we put:

Definition 1. (Modified Golden Rule) *In the current one-sector stochastic growth model, the modified Golden Rule k^* is determined by $f_1(k^*,A)=1$.*

Remark 1.

(i) By Assumption 2, it is easily noticed that $0<k^*<\infty$, i.e., the modified Golden Rule is a finite constant.

(ii) The FOC given by (2) is also a sufficient condition for the optimization of consumption due to $f_{11}<0$, which is given in Assumption 2.

Then, we can get:

Theorem 1. (Neighborhood Characterization of the Modified Golden Rule) *Based upon Proposition 1, we can get that there exists a constant* $0 < R < \infty$ *such that,*

$$\mu_\infty\left[\overline{B}_\beta(k^*)\right] \geq 1 - \frac{R}{\beta} \quad a.s.,$$

where $\overline{B}_\beta(k^*)$ *denotes the closure of,*

$$B_\beta(k^*) \triangleq \{k_t \in \mathbb{R}_+; |k_t - k^*| < \beta, t = 0,1,2,\ldots\},$$

for all $R < \beta < \infty$*.*

Proof. It follows from Proposition 1 that, $k_t < \infty$ a.s. for $t = 0,1,2,\ldots$, which combined with Remark 1 implies that there exists a constant $0 < R < \infty$ such that,

$$|k_t - k^*| \leq k_t + k^* \leq R \text{ a.s.}$$

Then, we have,

$$E_A\left[\frac{1}{T}\sum_{t=0}^{T-1}|k_t - k^*|\right] \leq R \text{ a.s.}$$

Hence,

$$\limsup_{T\to\infty} E_A\left[\frac{1}{T}\sum_{t=0}^{T-1}|k_t - k^*|\right] \leq R \text{ a.s.} \tag{3}$$

Define

$$B_\beta(k^*) \triangleq \{k_t \in \mathbb{R}_+ : |k_t - k^*| < \beta, t = 0,1,2,\ldots\},$$

and let $1_{\overline{B}^C_\beta(k^*)}(k_t)$ denote the indicator function of set $\overline{B}^C_\beta(k^*)$, the complementary set of $\overline{B}_\beta(k^*)$, then we obtain by (3),

$$\begin{aligned}
\mu_\infty\left[\overline{B}^C_\beta(k^*)\right] &= \limsup_{T\to\infty} E_A\left[\frac{1}{T}\sum_{t=0}^{T-1} 1_{\overline{B}^C_\beta(k^*)}(k_t)\right] \\
&\leq \limsup_{T\to\infty} E_A\left[\frac{1}{T}\sum_{t=0}^{T-1}\frac{|k_t - k^*|}{\beta}\right] \\
&\leq \frac{R}{\beta} \text{ a.s.,}
\end{aligned}$$

which yields

$$\mu_\infty\left[\overline{B}_\beta(k^*)\right] \geq 1 - \frac{R}{\beta} \text{ a.s.}$$

□

It follows from Proposition 2 that k_t also exhibits ergodic property, hence we can define the following Markov time,

$$\tau^* = \inf\{t \geq 0; k_t = k^*\}.$$

Hence, as in Dai (2012), we can establish:

Theorem 2. (Turnpike Theorem) *If k_t defines a martingale-path of capital accumulation and $\tau^* < \infty$ a.s., then we get that k_t strongly converges to k^* a.s. in the sense of uniform topology.*

Proof. By Doob's Martingale Inequality,

$$\mu_\infty\left(\sup_{0\leq t\leq T}|k_t| \geq \lambda\right) \leq \frac{1}{\lambda}E_A\left[|k_T|\right] = \frac{k_0}{\lambda}, \quad \forall\lambda > 0, \quad \forall T > 0.$$

Without loss of any generality, put $\lambda = 2^m$ for $m \in \mathbb{N}$, we get,

$$\mu_\infty\left(\sup_{0\leq t\leq T}|k_t| \geq 2^m\right) \leq \frac{k_0}{2^m}, \quad \forall T > 0.$$

Applying Borel-Cantelli Lemma reveals that,

$$\mu_\infty(\sup_{0\leq t\leq T}|k_t| \geq 2^m \text{ for infinitely many } m) = 0.$$

So, there is $\overline{m} \in \mathbb{N}$ such that,

$$\sup_{0\leq t\leq T}|k_t| < 2^m \text{ a.s. for } m \geq \overline{m}.$$

Hence,

$$\limsup_{T\to\infty}\sup_{0\leq t\leq T}|k_t| < 2^m \text{ a.s. for } m \geq \overline{m}. \tag{4}$$

Now, define

$$B_{2^{-m}}(\tau^*) \triangleq \left\{\tau \geq 0; |\tau - \tau^*| < 2^{-m}\right\}.$$

Thus, for $\forall\tau^m \in B_{2^{-m}}(\tau^*)$, and based on the assumption that $\tau^* < \infty$ a.s., by Doob's Optional Sampling Theorem and Doob's Martingale Inequality,

$$\mu_\infty\left(\sup_{0\leq t\leq\tau^m}|k_t - k^*| \geq \varepsilon\right) \leq \frac{1}{\varepsilon}E_A\left[|k_{\tau^m} - k^*|\right], \quad \forall\varepsilon > 0.$$

Due to (4), Remark 1 and the continuity of martingale w.r.t. time t, applying Lebesgue Bounded Convergence Theorem produces,

$$\limsup_{m\to\infty}\mu_\infty\left(\sup_{0\leq t\leq\tau^m}|k_t - k^*| \geq \varepsilon\right) \leq \frac{1}{\varepsilon}\limsup_{m\to\infty}E_A\left[|k_{\tau^m} - k^*|\right] = 0.$$

It follows from Fatou's Lemma that,

$$\mu_\infty\left(\sup_{0\le t\le\tau^*}|k_t-k^*|\ge\varepsilon\right)=0,$$

hence,

$$\mu_\infty\left(\sup_{0\le t\le\tau^*}|k_t-k^*|<\varepsilon\right)=1.$$

Thus, we obtain,

$$\sup_{0\le t\le\tau^*}|k_t-k^*|<\varepsilon \;\text{ a.s. }\; \forall\varepsilon>0.$$

By the arbitrariness of ε, we get,

$$\limsup_{\tau^*\to\infty}\sup_{0\le t\le\tau^*}|k_t-k^*|=0 \;\text{ a.s.}$$

□

Remark 2. Noting that the martingale path, in certain sense, can be regarded as the balanced path of capital accumulation for the stochastic growth models. It is hence confirmed by Theorem 2 that the *stochastic balanced path* of capital accumulation will converge to the modified Golden Rule given by Definition 1 in the long run and under relatively weak conditions, i.e., the modified Golden Rule is *reachable* in any almost surely finite time for the stochastic balanced path of capital accumulation. Accordingly, the asymptotic turnpike theorem holds true in the present stochastic case.

Furthermore, we can also derive,

Corollary 1. (Turnpike Theorem) *Combining Theorem 1 with Theorem 2 shows that for any given* $\varepsilon>0$,

$$\mu_\infty\left[\overline{B}_\beta(k^*)\right]\ge1-\frac{\varepsilon}{\beta}\approx1 \qquad a.s.,$$

where $\overline{B}_\beta(k^*)$ *denotes the closure of,*

$$B_\beta(k^*)\triangleq\{k_t\in\mathbb{R}_+;|k_t-k^*|<\beta,t=0,1,2,\ldots\}$$

for all$\varepsilon<\beta<\infty$.

Remark 3.

(i) The proof is quite similar to that of Theorem 1, so we omit it and leave it to the interested reader.

(ii) The economic intuition of Corollary 1 is that the ergodic Borel probability measure μ_∞ will place nearly all mass close to the modified Golden Rule. So, Corollary 1 can be regarded as the neighborhood type of turnpike theorems.

4. Concluding remarks

In the current investigation, we employ the basic model developed by Mirman (1972) and the technique pioneered by Dai (2012). We extend Mirman's stationary probability measure to an ergodic probability measure and define the stochastic version of balanced path of capital accumulation by the corresponding martingale path for the first time. Most importantly, corresponding turnpike theorems are demonstrated in relatively weak conditions. Noting that turnpike theorems have been always playing crucial roles in macroeconomics (e.g., Joshi 1997; McKenzie 1998; Dai 2012; Acemoglu 2012), we might argue that the present study indeed extends the classical turnpike theorems to much broader and stronger cases.

References

Acemoglu, D. (2012). Introduction to Economic Growth. *Journal of Economic Theory*, 147, 545–550.

Cass, D. (1972). On Capital Overaccumulation in the Aggregative, Neoclassical Model of Economic Growth: A Complete Characterization. *Journal of Economic Theory*, 4, 200–223.

Dai, D. (2012). Stochastic Versions of Turnpike Theorems in the Sense of Uniform Topology. *Annals of Economics and Finance*, 13, 389–431.

Joshi, S. (1997). Turnpike Theorem in Nonconvex and Nonstationary Environment. *International Economic Review*, 38, 225–248.

McKenzie, L. (1998). Turnpikes. *American Economic Review*, 88, 1–14.

Mirman, L. J. (1972). On the Existence of Steady State Measures for One Sector Growth Models with Uncertain Technology. *International Economic Review*, 13, 271–286.

Mitra, T. and Ray, D. (2012). On the Phelps-Koopmans Theorem. *Journal of Economic Theory*, 147, 833–849.

Phelps, E. S. (1961). The Golden Rule of Accumulation: A Fable for Growthmen. *American Economic Review*, 51, 638–643.

Phelps, E. S. (1962). The End of the Golden Age in Solovia: Comment. *American Economic Review*, 52, 1097–1099.

Phelps, E. S. (1965). Second Essay on the Golden Rule of Accumulation. *American Economic Review*, 55, 793–814.

Ray, D. (2009). The Phelps-Koopmans Theorem and Potential Optimality. *International Journal of Economic Theory*, 6, 11–28.

Samuelson, P. A. (1965). A Catenary Turnpike Theorem Involving Consumption and the Golden Rule. *American Economic Review*, 55, 486–496.