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Stable Bank Cooperation for Cost Reduction Problem

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Abstract In the paper we consider a problem of bank cost reduction by joint usage of ATMs. In coalition case, we assume that cooperation may be naturally restricted by a coalition structure. A question of stability of a coalition structure with respect to the Shapley value is investigated. Statements about the configuration of stable coalition structures are proposed. We also consider special cases in which their existence is proved.

Keywords Coalition structure, stability, cost allocation problem, Shapley value
JEL classification C71

1. Introduction

The problem of cost reduction is one of the natural problems of a company. In some cases companies (players in game theory) can reduce their costs using joint actions. In cooperative game theory it is suggested that players can form groups of players called coalitions. Players from the same coalition use joint actions in the game to maximize a coalition payoff. A cooperative game is usually determined by a characteristic function. The value of this function defined for a coalition can be interpreted as the profit or worth of the coalition that it can receive if the members of the coalition use joint actions maximizing the coalition payoff.

In classical cooperative game theory it is supposed that grand coalition is formed, but in many real problems players can be divided into some number of coalitions, and this division leads to a game with coalition structure. That is why, players can benefit not from being a member of the grand coalition but from being a member of a smaller one. Games with coalition structures are applicable to the problems in politics, economics, where mostly grand coalition cannot be formed because of many reasons. They are the so-called cooperative games with restricted cooperation. Some ideas of solution concepts in this class of games are examined by Aumann and Drèze (1974), Faigle and Kern (1992), Katsev and Yanovskaya (2013), and Naumova (2012).

In the paper, we consider a modification of the model proposed by Bjorndal et al. (2004) applicable to the cost reduction problem for banks by joint usage of ATMs. The

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modification of the model changes the characteristic function, which becomes non-superadditive in general case.

The next natural problem in the theory of games with coalition structures is how to choose stable in some sense coalition structure among all possible coalition structures. We also try to answer this question and suggest an approach to define a stable coalition structure. We demand a stable coalition structure to satisfy the property of individual rationality. As a payoff function of a player we use his component in an allocation for some cooperative solution concept. The existence of stable coalition structure with respect to the Shapley value and the equal surplus solution for the cases of two- and three-person games was proved in Sedakov et al. (2013). Similar stability concepts were used in Haeringer (2001), Hart and Kurz (1983) and Tiebout (1956) for general payoff functions in a strategic (non-cooperative) setting. Bogomolnaia and Jackson (2002) also investigate this concept for additively separable and symmetric payoff functions.

In Parilina (2007) a two-stage game of cost allocation among companies using the joint resources is considered. In Gow and Thomas (1998) and Nouweland et al. (1996) the model of fund transfers among banks is introduced. In the paper, we find a stable in some sense coalition structure in the non-dynamic case, although in theory of games with coalition structures the problem of coalition formation is also studied for a dynamic case. In particular, in Petrosjan and Mamkina (2006) and Petrosyan et al. (2006) different approaches of coalition formation in dynamic case are proposed. Authors consider dynamic games with prescribed so-called coalitional function, games with stochastic nature of coalition formation, and games with restricted coalition formation. In Bloch (1996), the coalition structure is supposed to be formed consequently by players. Unfortunately, all these studies do not deal with a problem of coalition structures stability.

The results of the paper can be shortly described as follows. In the paper we consider the game with the characteristic function representing the bank costs that can be saved if they consolidate their ATMs in a common network. It is shown that the characteristic function may be non-superadditive in some cases. We examine the stability concept of coalition structures which is similar to the concept of Nash equilibrium for noncooperative strategic form games. We prove that if coalition structure is stable, then the payment distribution is an allocation for which the individual rationality condition is satisfied.

Special cases when only one or two banks have ATMs are examined, and the configuration of stable coalition structures with respect to the Shapley value is obtained. We also consider the case when some banks have ATMs and some do not. Here with the help of proved propositions we can a priori determine and exclude some unstable coalition structures.

The paper is organized as follows. In Section 2, we consider the problem statement, provide the expression of the characteristic function, and show that the characteristic function is non-superadditive in general. In Section 3, we introduce a game with coalition structure. Expression of the Shapley value which is chosen as a cooperative solution concept, and stability concepts of coalition structures with respect to the cooperative solution concept are introduced. In Section 4, we investigate a problem of stability of coalition structures with respect to the Shapley value. In particular cases we find stable coalition structures.

2. The model

Let N be a finite set of banks which operate in some region. In this region banks are allowed to place ATMs, and bank clients can use them to withdraw cash. It is supposed that if a bank has ATMs in the region, its clients use only ATM to withdraw cash. Two or more banks may consolidate their ATMs in one network, and in this case bank clients use ATMs of the network to withdraw cash with equal probabilities.

Bank transaction costs are equal to $\alpha > 0$ if a bank client uses the ATM of his own bank for cash withdrawal. If the bank client uses the ATM of another bank which belongs to the same ATM network with his own bank, bank transaction costs are equal to $\beta > \alpha$. In other cases, bank transaction costs are equal to $\gamma > \beta$. These can be the case of ATM absence in the region, or non-ATM method of cash withdrawal etc. We suppose parameters α, β and γ are equal for all banks. All the costs α, β, and γ are bank costs. In the paper we are interested in reducing bank costs, and do not take into account client costs for cash withdrawal.

A bank $i \in N$ has two numerical characteristics: $n_i > 0$ representing a number of bank transactions, and $k_i \geq 0$ representing a number of bank ATMs placed in the region.

We understand a coalition S as a non-empty subset of banks of the set N, which consolidate their ATMs in a common network. For each coalition S, a number $k(S) = \sum_{i \in S} k_i$ represents the total number of ATMs of banks from S in the region.

Denote a set of banks which have ATMs in the region by $A \subseteq N$. For any bank $i \in N$ its costs for clients' transactions have the form:

$$c(\{i\}) = \begin{cases} \alpha n_i, & \text{if } i \in A, \\ \gamma n_i, & \text{otherwise.} \end{cases}$$

For any non-empty coalition S, total bank costs for clients' transactions are:

$$c(S) = \begin{cases} \alpha \sum\limits_{i \in S} \dfrac{k_i}{k(S)} n_i + \beta \sum\limits_{i \in S} \left(1 - \dfrac{k_i}{k(S)}\right) n_i, & \text{if } S \cap A \neq \emptyset, \\ \gamma \sum\limits_{i \in S} n_i, & \text{if } S \cap A = \emptyset. \end{cases} \tag{1}$$

In case $S \cap A \neq \emptyset$, the first term in (1) is total expected costs of banks from S if their clients use original ATMs. The second term in the expression represents total expected costs of banks from S if their clients use ATMs of another banks from S. Here we assume that a client of the bank uses either the original ATM or other ATMs from the common network with the equal probabilities. This assumption is natural in the modern bank industry where newly formed banks prefer not to operate their own ATMs but use the ATMs of other banks paying them for all served clients. Moreover, in this case clients do not pay any extra fees from their withdrawal.

Using (1) we can consider cost-saving cooperative game (N,v) with a characteristic function v defined for each $S \subseteq N$ as follows:

$$\begin{aligned} v(S) &= \sum_{i\in S} c(\{i\}) - c(S) = &(2)\\ &= \begin{cases} (\gamma-\beta)\sum_{i\in S\setminus A} n_i - (\beta-\alpha)\sum_{i\in S\cap A}\left(1-\frac{k_i}{k(S)}\right)n_i, & \text{if } S\cap A \neq \emptyset,\\ 0, & \text{if } S\cap A = \emptyset. \end{cases} \end{aligned}$$

Value $v(S), S \subseteq N$ means the costs which banks from S can save if they consolidate their ATMs in a common network.

If $S\cap A = \emptyset$, it is obvious that $v(S) = 0$. If $S\cap A \neq \emptyset$ following (1), we have:

$$\begin{aligned} v(S) &= \alpha\sum_{i\in S\cap A} n_i + \gamma\sum_{i\in S\setminus A} n_i - \alpha\sum_{i\in S}\frac{k_i}{k(S)}n_i - \beta\sum_{i\in S}\left(1-\frac{k_i}{k(S)}\right)n_i\\ &= \alpha\sum_{i\in S\cap A} n_i + \gamma\sum_{i\in S\setminus A} n_i - \alpha\sum_{i\in S\cap A}\frac{k_i}{k(S)}n_i - \alpha\sum_{i\in S\setminus A}\frac{k_i}{k(S)}n_i\\ &\quad -\beta\sum_{i\in S\cap A}\left(1-\frac{k_i}{k(S)}\right)n_i - \beta\sum_{i\in S\setminus A}\left(1-\frac{k_i}{k(S)}\right)n_i\\ &= (\gamma-\beta)\sum_{i\in S\setminus A} n_i - (\beta-\alpha)\sum_{i\in S\cap A}\left(1-\frac{k_i}{k(S)}\right)n_i\\ &\quad +(\beta-\alpha)\sum_{i\in S\setminus A}\frac{k_i}{k(S)}n_i. \end{aligned}$$

The last summand $(\beta-\alpha)\sum_{i\in S\setminus A}\frac{k_i}{k(S)}n_i$ is equal to zero since all $k_i = 0$, $i \in S\setminus A$, i.e. banks from $S\setminus A$ have no ATMs. Note that $v(\{i\}) = 0$ for all $i \in N$.

The characteristic function v defined by (2) does not satisfy the superadditivity condition, i.e. for any two non-empty disjoint coalitions $S \subset N$ and $T \subset N$ the inequality $v(S\cup T) \geq v(S) + v(T)$ does not hold in general. To prove that the function v is not superadditive, it is sufficient to consider the case $S = \{i\}$, $T = \{j\}$, and $i, j \in A$. In this case

$$v(\{i,j\}) - v(\{i\}) - v(\{j\}) = -(\beta-\alpha)\cdot\frac{k_j n_i + k_i n_j}{k_i + k_j} < 0.$$

3. Coalition game

Since the superadditivity condition of the characteristic function v defined by (2) does not hold in general, it makes sense to consider a game with coalition structure.

Definition 1. *Coalition structure π is a partition $\{B_1,\ldots,B_m\}$ of the set N, i.e. $B_1\cup\ldots\cup B_m = N$, and $B_i\cap B_j = \emptyset$ for all $i, j = 1,\ldots,m, i\neq j$.*

Denote a game with the player set N, characteristic function v defined by (2), and coalition structure π by (N, v, π).

Definition 2. *A profile $x^\pi = (x_1^\pi, \dots, x_n^\pi) \in \mathbb{R}^n$ is a payment distribution in the game (N, v, π) with coalition structure π if the efficiency condition, i.e. $\sum_{i \in B_j} x_i^\pi = v(B_j)$ holds for all coalitions $B_j \in \pi$.*

Definition 3. *A payment distribution x^π is an allocation in the game (N, v, π) with coalition structure π if the individual rationality condition, i. e. $x_i^\pi \geq v(\{i\})$ holds for any player $i \in N$.*

Applying Definition 3 to our model, we obtain the individual rationality condition as follows: $x_i^\pi \geq 0$ for all players from N.

Denote the coalition structure $\pi \setminus B_i \subset \pi$ by π_{-B_i}, and the coalition which contains player $i \in N$ by $B(i) \in \pi$.

The key role in the theory of games with coalition structure is a solution concept. All of them are different by its nature, and before the game starts, players have to choose a solution concept and use it for cost allocation. Once the solution concept has been chosen, it is not reviewed and remains fixed. Since we deal with coalition structure, one needs to select an efficient, single-valued cooperative solution concept that would take into account players' contributions to different coalitions. In the paper we consider the Shapley value (Shapley 1953) as a solution concept for games with coalition structure. For a game (N, v, π), $\pi = \{B_1, \dots, B_m\}$, the component of the Shapley value ϕ_i^π, $i \in N$ is calculated as follows:

$$\phi_i^\pi = \sum_{S \subseteq B(i), i \in S} \frac{(|B(i)| - |S|)!(|S| - 1)!}{|B(i)|!} [v(S) - v(S \setminus \{i\})]. \tag{3}$$

One may consider nucleolus, the equal surplus solution (Driessen and Funaki 1991; Schmeidler 1969) as other single-valued cooperative solution concepts.

In cooperative game theory there are different approaches to the stability concept of the coalition structure suggested by Haeringer (2001), Hart and Kurz (1983), Marini (2009), and Tiebout (1956). In the present study, we propose an approach similar to the Nash equilibrium concept which guarantees players to stay in their coalitions and prevents revision of the coalition structure.

Definition 4. *Coalition structure $\pi = \{B_1, \dots, B_m\}$ is said to be stable with respect to a single-valued cooperative solution concept if for any player $i \in N$ the inequality*

$$x_i^\pi \geq x_i^{\pi'} \text{ holds for all } \pi' = \{B(i) \setminus \{i\}, B_j \cup \{i\}, \pi_{-B(i) \cup B_j}\},$$

where $B_j \in \pi \cup \emptyset$, and $B_j \neq B(i)$. Here x^π and $x^{\pi'}$ are two payment distributions calculated according to the chosen cooperative solution concept for games (N, v, π) and (N, v, π') with coalition structures.

There is an obvious similarity between the Nash equilibrium concept and coalition structure stability concept from Definition 4. Here player's strategies are to join any existing coalition or to play as an individual player. But if the player deviates from the coalition to which he belongs in so-called stable coalition structure, provided that all other players follow their coalitions, he never benefits (saves more costs).

In Definition 4 we make one assumption: if player $i \in B(i)$ leaves coalition $B(i)$, coalition $B(i) \setminus \{i\}$ does not break, and is still the part of the coalition structure. Therefore, player i can join any existing coalition in the current coalition structure without any restrictions.

Proposition 1. *If coalition structure π is stable with respect to the chosen cooperative solution, then payment distribution x^π is an allocation.*

Proof. We use method ex adverso to prove the statement. Suppose that coalition structure π is stable with respect to the cooperative solution, and payment distribution x^π, calculated according to the cooperative solution for π is not an allocation. It means the existence of coalition $B \in \pi$, $|B| > 1$, such that inequality $x_i^\pi < v(\{i\})$ holds at least for one player $i \in B$. If player i deviates from coalition B becoming a singleton $\{i\}$, coalition structure π is replaced by coalition structure $\pi' = \{\{i\}, \pi_{-\{i\}}\}$, and player i benefits: $x_i^{\pi'} = v(\{i\})$. Therefore, by Definition 4 coalition structure π is not stable with respect to the chosen cooperative solution. This contradiction proves the statement. □

With the help of Proposition 1 we conclude that if for a coalition structure the payment distribution is an allocation, then the coalition structure may be stable, otherwise, it is always unstable.

4. Stable coalition structures with respect to the Shapley value

4.1 General case: $A \subset N$, $|A| > 1$

Suppose there are more than one player with ATMs, i.e. $|A| > 1$. If there is a coalition $B_j \subseteq A$ in coalition structure $\pi = \{B_1, \ldots, B_m\}$, then, naturally, π is unstable with respect to any cooperative solution concept. It follows from the inequality $v(B_j) < 0$ according to (2). It means that there exists at least one player $i \in B_j$ whose component x_i^π in payment distribution $x^\pi = (x_1^\pi, \ldots, x_n^\pi)$ is negative ($x_i^\pi < 0$) with respect to any chosen cooperative solution. Therefore, he can deviate from the current structure π to increase his payoff up to zero becoming an individual player.

Proposition 2. *Let set A in a game (N, v, π) be non-empty, and coalition structure π contain a set B_j such that $B_j \cap A = \emptyset$. Then coalition structure π is unstable with respect to the Shapley value.*

Proof. In coalition structure π there exists a coalition B_j such that all banks belonging to this coalition do not have ATMs in the considered region. But among banks from N there exists at least one bank $m \in A \cap N$ which owns ATMs. Without loss of generality, suppose player m belongs to coalition $B(m)$ from coalition structure π.

Consider any player i belonging to coalition B_j. Obviously, his Shapley value component ϕ_i^π is equal to zero. Let player i deviate from coalition B_j and join coalition $B(m)$. Therefore, the new coalition structure $\pi' = \{B_j \setminus \{i\}, B(m) \cup \{i\}, \pi_{-B(m) \cup B_j}\}$ is realized. Prove that $\phi_i^{\pi'} > \phi_i^{\pi} = 0$, i.e. player i can increase his Shapley value component forming coalition structure π'.

The Shapley value component of player i can be calculated by equation (3) summing up over possible coalitions $S \subseteq B(m) \cup i$, $S \ni i$. Coalition S is one of the two types:

(i) $S \cap A = \emptyset$, in this case $v(S) - v(S \setminus \{i\}) = 0$.

(ii) $S \cap A \neq \emptyset$, in this case $v(S) - v(S \setminus \{i\}) = (\gamma - \beta)n_i > 0$.

Then following formula (3) we obtain: $\phi_i^{\pi'} > 0 = \phi_i^{\pi}$. As coalition structure π' satisfies the conditions of Definition 4, then coalition structure π considered in Proposition 2 is not stable with respect to the Shapley value. □

Corollary 1. *Suppose that in a game with coalition structure (N, v, π) set A is nonempty and does not coincide with N, and coalition structure $\pi = \{B_1, \ldots, B_m\}$ is stable with respect to the Shapley value. Then $B_i \cap A \neq \emptyset$ for any $i = 1, \ldots, m$. Specifically, under described conditions the number of coalitions in structure π does not exceed the cardinality of set A: $m \leq |A|$.*

Example 1. Consider a five-player game: $N = \{1,2,3,4,5\}$. Let the set of banks owned ATMs be $A = \{1,2,3\}$ and the set of banks without ATMs be $N \setminus A = \{4,5\}$. The number of ATMs that banks own are $k_1 = 10$, $k_2 = 5$, $k_3 = 2$, $k_4 = k_5 = 0$. In the example the following parameters are: costs per a client's service $\alpha = 0.3$, $\beta = 0.8$, $\gamma = 2$, the number of bank transactions are $n_1 = 10{,}000$, $n_2 = 7{,}000$, $n_3 = 2{,}000$, $n_4 = 6{,}000$, $n_5 = 1{,}000$.

Applying propositions from this Section, we find only 4 out of 52 stable coalition structures with respect to the Shapley value in terms of Definition 4. The stable coalition structures are $\{\{1\},\{2,3,4,5\}\}$, $\{\{1,3,4,5\},\{2\}\}$, $\{\{1\},\{2,4\},\{3,5\}\}$ and $\{\{1,4\},\{2\},\{3,5\}\}$ with the following allocations are introduced in Table 1.

Table 1. Stable coalition structures in Example 1

π	ϕ_1^{π}	ϕ_2^{π}	ϕ_3^{π}	ϕ_4^{π}	ϕ_5^{π}
$\{\{1\},\{2,3,4,5\}\}$	0	542.86	542.86	4,800	800
$\{\{1,3,4,5\},\{2\}\}$	566.67	0	566.67	4,800	800
$\{\{1\},\{2,4\},\{3,5\}\}$	0	3,600	600	3,600	600
$\{\{1,4\},\{2\},\{3,5\}\}$	3,600	0	600	3,600	600

All 52 possible coalition structures are shown in Table 2, in which four stable coalition structures with respect to the Shapley value are marked in bold.

4.2 Only one bank owns ATMs: $|A| = 1$

Supposing that $N = \{1, \ldots, n\}$ is the set of banks, let only bank 1 own $k_1 > 0$ ATMs, $k_i = 0$, $i = 2, \ldots, n$, i. e. $A = \{1\}$. Using expression (2) we can rewrite the expressions

Table 2. The Shapley values for all coalition structures

π	ϕ_1^π	ϕ_2^π	ϕ_3^π	ϕ_4^π	ϕ_5^π
$\{\{1,2,3,4,5\}\}$	−1,476.94	−1,500.75	−334.08	5,400	900
$\{\{\mathbf{1}\},\{\mathbf{2,3,4,5}\}\}$	0	542.86	542.86	4,800	800
$\{\{1,2,3,4\},\{5\}\}$	−1,576.94	−1,600.75	−434.08	5,400	0
$\{\{1,2,3,5\},\{4\}\}$	−2,076.94	−2,100.75	−934.08	0	900
$\{\{1,2,4,5\},\{3\}\}$	−600	−600	0	4,800	800
$\{\{\mathbf{1,3,4,5}\},\{\mathbf{2}\}\}$	566.67	0	566.67	4,800	800
$\{\{1,2\},\{3,4,5\}\}$	−2,000	−2,000	4,200	3,600	600
$\{\{1,3\},\{2,4,5\}\}$	−833.33	4,200	−833.33	3,600	600
$\{\{1,4\},\{2,3,5\}\}$	3,600	−657.14	−657.14	3,600	800
$\{\{1,5\},\{2,3,4\}\}$	600	342.86	342.86	4,800	600
$\{\{1,2,3\},\{4,5\}\}$	−2,176.94	−2,200.75	−1,034.08	0	0
$\{\{1,2,4\},\{3,5\}\}$	−800	−800	600	4,800	600
$\{\{1,2,5\},\{3,4\}\}$	−1,800	−1,800	3,600	3,600	800
$\{\{1,3,4\},\{2,5\}\}$	366.67	600	366.67	4,800	600
$\{\{1,3,5\},\{2,4\}\}$	−633.33	3,600	−633.33	3,600	800
$\{\{1,4,5\},\{2,3\}\}$	4,200	−857.14	−857.14	3,600	600
$\{\{1\},\{2\},\{3,4,5\}\}$	0	0	4,200	3,600	600
$\{\{1\},\{2,3,4\},\{5\}\}$	0	342.86	342.86	4,800	0
$\{\{1\},\{2,3,5\},\{4\}\}$	0	−657.14	−657.14	0	800
$\{\{1\},\{2,4,5\},\{3\}\}$	0	4,200	0	3,600	600
$\{\{1,2,3\},\{4\},\{5\}\}$	−2,176.94	−2,200.75	−1,034.08	0	0
$\{\{1,2,4\},\{3\},\{5\}\}$	−800	−800	0	4,800	0
$\{\{1,2,5\},\{3\},\{4\}\}$	−1,800	−1,800	0	0	800
$\{\{1,3,4\},\{2\},\{5\}\}$	366.67	0	366.67	4,800	0
$\{\{1,3,5\},\{2\},\{4\}\}$	−633.33	0	−633.33	0	800
$\{\{1,4,5\},\{2\},\{3\}\}$	4,200	0	0	3,600	600
$\{\{1\},\{2,3\},\{4,5\}\}$	0	−857.14	−857.14	0	0
$\{\{\mathbf{1}\},\{\mathbf{2,4}\},\{\mathbf{3,5}\}\}$	0	3,600	600	3,600	600
$\{\{1\},\{2,5\},\{3,4\}\}$	0	600	3,600	3,600	600
$\{\{1,2\},\{3\},\{4,5\}\}$	−2,000	−2,000	0	0	0
$\{\{1,2\},\{3,4\},\{5\}\}$	−2,000	−2,000	3,600	3,600	0
$\{\{1,2\},\{3,5\},\{4\}\}$	−2,000	−2,000	600	0	600
$\{\{1,3\},\{2\},\{4,5\}\}$	−833.33	0	−833.33	0	0
$\{\{1,3\},\{2,4\},\{5\}\}$	−833.33	3,600	−833.33	3,600	0
$\{\{1,3\},\{2,5\},\{4\}\}$	−833.33	600	−833.33	0	600
$\{\{\mathbf{1,4}\},\{\mathbf{2}\},\{\mathbf{3,5}\}\}$	3,600	0	600	3,600	600
$\{\{1,4\},\{2,3\},\{5\}\}$	3,600	−857.14	−857.14	3,600	0
$\{\{1,4\},\{2,5\},\{3\}\}$	3,600	600	0	3,600	600
$\{\{1,5\},\{2\},\{3,4\}\}$	600	0	3,600	3,600	600
$\{\{1,5\},\{2,3\},\{4\}\}$	600	−857.14	−857.14	0	600
$\{\{1,5\},\{2,4\},\{3\}\}$	600	3,600	0	3,600	600
$\{\{1\},\{2\},\{3\},\{4,5\}\}$	0	0	0	0	0
$\{\{1\},\{2\},\{3,4\},\{5\}\}$	0	0	3,600	3,600	0
$\{\{1\},\{2\},\{3,5\},\{4\}\}$	0	0	600	0	600
$\{\{1\},\{2,3\},\{4\},\{5\}\}$	0	−857.14	−857.14	0	0
$\{\{1\},\{2,4\},\{3\},\{5\}\}$	0	3,600	0	3,600	0
$\{\{1\},\{2,5\},\{3\},\{4\}\}$	0	600	0	0	600
$\{\{1,2\},\{3\},\{4\},\{5\}\}$	−2,000	−2,000	0	0	0
$\{\{1,3\},\{2\},\{4\},\{5\}\}$	−833.33	0	−833.33	0	0
$\{\{1,4\},\{2\},\{3\},\{5\}\}$	3,600	0	0	3,600	0
$\{\{1,5\},\{2\},\{3\},\{4\}\}$	600	0	0	0	600
$\{\{1\},\{2\},\{3\},\{4\},\{5\}\}$	0	0	0	0	0

of the characteristic function for a coalition $S \subseteq N$:

$$v(S) = \begin{cases} (\gamma-\beta) \sum\limits_{i \in S \setminus \{1\}} n_i, & 1 \in S, \\ 0, & 1 \notin S, \end{cases} \tag{4}$$

and calculate the component ϕ_i of the Shapley value for player $i \neq 1$. Notice that

$$v(S) - v(S \setminus \{i\}) = \begin{cases} (\gamma-\beta) n_i, & 1 \in S, \\ 0, & 1 \notin S. \end{cases}$$

Therefore, for calculation of the Shapley value component ϕ_i, $i = 2, \ldots, n$ it is sufficient to consider only coalitions containing both players i and 1. We obtain the expression:

$$\begin{aligned} \phi_i &= \sum_{S \subseteq N, i \in S, 1 \in S} \frac{(|N|-|S|)!(|S|-1)!}{|N|!} (\gamma-\beta) n_i \\ &= (\gamma-\beta) n_i \sum_{s=2}^{n} \frac{(n-s)!(s-1)!}{n!} \cdot \binom{n-2}{s-2} = \frac{(\gamma-\beta) n_i}{2}. \end{aligned} \tag{5}$$

Since the Shapley value is the efficient payment distribution, then component ϕ_1 of the Shapley value for player 1 has the form:

$$\phi_1 = v(N) - \sum_{i=2}^{n} \phi_i = (\gamma-\beta) \sum_{i=2}^{n} n_i - \sum_{i=2}^{n} \frac{(\gamma-\beta) n_i}{2} = \frac{(\gamma-\beta)}{2} \sum_{i=2}^{n} n_i. \tag{6}$$

Here we notice that the Shapley value components from equations (5) and (6) are always positive.

Proposition 3. *In a game $(N, v, \overline{\pi})$ in case $|A| = 1$, i.e. when only one bank owns ATMs, characteristic function* (4) *is superadditive. Moreover, there exists a unique stable coalition structure $\overline{\pi} = \{N\}$ with respect to the Shapley value, and the Shapley value components are determined by expressions* (5) *and* (6).

Proof. At first, we prove the superadditivity of function (4). Choose any two non-empty disjoint coalitions S and T. Since the only bank 1 owns ATMs, then without loss of generality we have only two cases: either $1 \in S$, $1 \notin T$ or $1 \notin S$, $1 \notin T$. In the first case we have:

$$\begin{aligned} & v(S \cup T) - v(S) - v(T) \\ & \qquad = (\gamma-\beta) \sum_{i \in (S \cup T) \setminus \{1\}} n_i - (\gamma-\beta) \sum_{i \in S \setminus \{1\}} n_i = (\gamma-\beta) \sum_{i \in T} n_i > 0. \end{aligned}$$

In the second case $v(S \cup T) - v(S) - v(T) = 0$. Thus, the superadditivity condition of the characteristic function in case $|A| = 1$ is proved.

Now prove stability of coalition structure $\overline{\pi} = \{N\}$ with respect to the Shapley value which components $\phi_1^{\overline{\pi}}$ and $\phi_i^{\overline{\pi}}$, $i = 2, \ldots, n$ are calculated by (5) and (6). If player 1 leaves coalition N and becomes an individual player, then in coalition structure $\pi' = \{\{1\}, N \setminus \{1\}\}$ his component of the Shapley value $\phi_1^{\pi'}$ is equal to zero. If any other player $i \neq 1$ leaves coalition N and becomes an individual player, then according to the new coalition structure $\pi'' = \{\{i\}, N \setminus \{i\}\}$ his Shapley value component $\phi_i^{\pi''}$ is also equal to zero. Therefore, following Definition 4 coalition structure $\overline{\pi} = \{N\}$ is stable with respect to the Shapley value.

To prove the uniqueness of stable coalition structure $\overline{\pi} = \{N\}$ consider any coalition structure $\tilde{\pi} = \{B_1, \ldots, B_m\} \neq \overline{\pi}$. Without loss of generality suggest that $1 \in B_1$. All players from sets $B_2, \ldots, B_m$ according to expression (2) have zero Shapley value components. Obviously, there exists a player $i \neq 1$ such that $i \notin B_1$. If player i deviates from coalition structure $\tilde{\pi}$ and joins coalition B_1, then his component of the Shapley value becomes equal to $(\gamma - \beta)n_i/2 > 0$. As we choose coalition structure and player i at random, we can state that any coalition structure different from $\overline{\pi}$ (i. e. structure which contains at least one player from set $N \setminus 1$ and he does not belong to coalition containing player 1) is not stable with respect to the Shapley value following Definition 4. Hence, a unique stable coalition structure in case $|A| = 1$ is $\overline{\pi} = \{N\}$. □

Example 2. Consider again a five-player game. Let the set of banks owned ATMs be $A = \{1\}$ and the set of banks without ATMs be $N \setminus A = \{2, 3, 4, 5\}$. The number of ATMs that banks own are $k_1 = 10$, $k_2 = k_3 = k_4 = k_5 = 0$. The parameters of the game are: costs per a client's service $\alpha = 0.3$, $\beta = 0.8$, $\gamma = 2$, the number of bank transactions are $n_1 = 10{,}000$, $n_2 = 7{,}000$, $n_3 = 2{,}000$, $n_4 = 6{,}000$, $n_5 = 1{,}000$.

Following Proposition 3, there is a unique stable coalition structure with respect to the Shapley value $\overline{\pi} = \{\{1, 2, 3, 4, 5\}\}$ and its components are (see (5) and (6)):

$$\phi_1^{\overline{\pi}} = 9{,}600,\ \phi_2^{\overline{\pi}} = 4{,}200,\ \phi_3^{\overline{\pi}} = 1{,}200,\ \phi_4^{\overline{\pi}} = 3{,}600,\ \phi_5^{\overline{\pi}} = 600.$$

4.3 Two banks own ATMs: $|A| = 2$

Let $N = \{1, \ldots, n\}$ be the set of banks. Here two banks 1 and 2 own $k_1 > 0$ and $k_2 > 0$ ATMs respectively, and $k_i = 0$, $i = 3, \ldots, n$. In this case the set $A = \{1, 2\}$. Again, using expression (2) and denoting $\delta = \frac{\beta - \alpha}{\gamma - \beta} \cdot \frac{k_2 n_1 + k_1 n_2}{k_1 + k_2}$ we obtain:

$$v(S) = \begin{cases} (\gamma - \beta) \sum\limits_{i \in S \setminus \{1\}} n_i, & 1 \in S, 2 \notin S, \\ (\gamma - \beta) \sum\limits_{i \in S \setminus \{2\}} n_i, & 1 \notin S, 2 \in S, \\ (\gamma - \beta) \left[\sum\limits_{i \in S \setminus \{1,2\}} n_i - \delta \right], & 1 \in S, 2 \in S, \\ 0, & 1 \notin S, 2 \notin S. \end{cases} \tag{7}$$

To calculate the component ϕ_i of the Shapley value for player $i = 3, \ldots, n$ notice that

$$v(S) - v(S \setminus \{i\}) = \begin{cases} 0, & 1 \notin S, 2 \notin S, \\ (\gamma - \beta) n_i, & \text{otherwise.} \end{cases}$$

Therefore, for calculation of the Shapley value component ϕ_i, $i=3,\dots,n$ one needs to consider only coalitions containing player i and at least one player from A. We obtain the expression:

$$\begin{aligned}\phi_i &= (\gamma-\beta)n_i\left[\sum_{s=3}^{n}\frac{(n-s)!(s-1)!}{n!}\cdot\binom{n-3}{s-3}+2\sum_{s=2}^{n-1}\frac{(n-s)!(s-1)!}{n!}\cdot\binom{n-3}{s-2}\right]\\ &= \frac{2(\gamma-\beta)n_i}{3}. \end{aligned} \tag{8}$$

Notice that $\phi_i>0$, $i=3,\dots,n$.

Consider player 1. For all $S\subseteq N$ and $1\in S$ we have

$$v(S)-v(S\setminus\{1\})=\begin{cases}(\gamma-\beta)\sum\limits_{i\in S\setminus\{1\}}n_i, & 1\in S, 2\notin S,\\ -(\gamma-\beta)\delta, & 1\in S, 2\in S,\\ 0, & 1\notin S.\end{cases} \tag{9}$$

For player 2 we have

$$v(S)-v(S\setminus\{2\})=\begin{cases}(\gamma-\beta)\sum\limits_{i\in S\setminus\{2\}}n_i, & 1\notin S, 2\in S,\\ -(\gamma-\beta)\delta, & 1\in S, 2\in S,\\ 0, & 2\notin S.\end{cases} \tag{10}$$

Comparing marginal contributions (9) and (10), we conclude that $\phi_1=\phi_2$. Since the Shapley value is the efficient payment distribution, components ϕ_1 and ϕ_2 are equal to:

$$\begin{aligned}\phi_1=\phi_2 &= \frac{1}{2}\left[v(N)-\sum_{i=3}^{n}\phi_i\right]=\frac{1}{2}\left[(\gamma-\beta)\left[\sum_{i=3}^{n}n_i-\delta\right]-\frac{2}{3}(\gamma-\beta)\sum_{i=3}^{n}n_i\right]\\ &= \frac{(\gamma-\beta)}{6}\left[\sum_{i=3}^{n}n_i-3\delta\right]. \end{aligned} \tag{11}$$

Proposition 4. *In a game (N,v,π) in case $A=\{1,2\}$ there exists stable coalition structure with respect to the Shapley value. Moreover, if coalition structure $\pi_1=\{\{1\},N\setminus\{1\}\}$ is stable with respect to the Shapley value, coalition structure $\pi_2=\{\{2\},N\setminus\{2\}\}$ is also stable with respect to the Shapley value and vice versa.*

Proof. Consider coalition structure $\pi=\{B_1,B_2\}$, $1\in B_1$, $2\in B_2$, $|B_1|>1$, $|B_2|>1$. If π is stable with respect to the Shapley value, the following inequalities must hold:

$$\begin{cases}3\sum\limits_{i\in B_1\setminus\{1\}}n_i\geq\sum\limits_{i\in B_2\setminus\{2\}}n_i-3\delta,\\ 3\sum\limits_{i\in B_2\setminus\{2\}}n_i\geq\sum\limits_{i\in B_1\setminus\{1\}}n_i-3\delta.\end{cases} \tag{12}$$

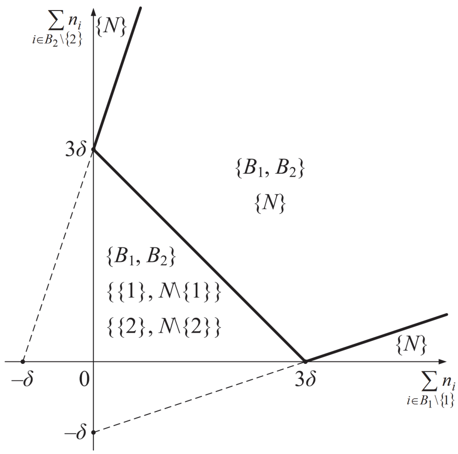

Figure 1. Stable coalition structures in case $|A| = 2$

Coalition structure π_1 is stable with respect to the Shapley value iff $\sum_{i\in N\setminus\{1,2\}} n_i \le 3\delta$. Under the same condition coalition structure π_2 will also be stable with respect to the Shapley value. Coalition structure $\{N\}$ is stable with respect to the Shapley value iff $\sum_{i\in N\setminus\{1,2\}} n_i \ge 3\delta$. Coalition structure π_i is unstable with respect to the Shapley value since at least player i joining coalition $N\setminus\{i\}$ gets more.

In Figure 1 we can see all possible stable coalition structures with respect to the Shapley value which depend on the values $\sum_{i\in B_1\setminus\{1\}} n_i$ and $\sum_{i\in B_2\setminus\{2\}} n_i$. $\square$

According to Corollary 1, coalition structure $\pi_i = \{\{i\}, N\setminus\{i\}\}$, $i\in N\setminus A$ is always unstable with respect to the Shapley value.

Example 3. Consider a five-player game. Let the set of banks owned ATMs be $A = \{1,2\}$ and the set of banks without ATMs be $N\setminus A = \{3,4,5\}$. The number of ATMs that banks own are $k_1 = 10$, $k_2 = 5$, $k_3 = k_4 = k_5 = 0$. The parameters are: costs per a client's service $\alpha = 0.3$, $\beta = 0.8$, $\gamma = 2$, the number of bank transactions are $n_1 = 10{,}000$, $n_2 = 7{,}000$, $n_3 = 2{,}000$, $n_4 = 6{,}000$, $n_5 = 1{,}000$.

Following Corollary 1 and Proposition 4, we get 8 out of 52 stable coalition structure with respect to the Shapley value, and the corresponding allocations (see expressions (8) and (11)) are demonstrated in Table 3.

4.4 All banks own ATMs: $A = N$

Let set $N = \{1,\dots,n\}$ be the set of banks such that bank $i\in N$ owns $k_i > 0$ ATMs in the considered region. Using expression (2) we can calculate characteristic function for any coalition $S\subseteq N$:

$$v(S) = -(\beta-\alpha)\sum_{i\in S}\left(1-\frac{k_i}{k(S)}\right)n_i. \tag{13}$$

Table 3. Stable coalition structures in Example 3

π	ϕ_1^π	ϕ_2^π	ϕ_3^π	ϕ_4^π	ϕ_5^π
$\{\{1\},\{2,3,4,5\}\}$	0	5,400	1,200	3,600	600
$\{\{1,3,4\},\{2,5\}\}$	4,800	600	1,200	3,600	600
$\{\{1,3,4,5\},\{2\}\}$	5,400	0	1,200	3,600	600
$\{\{1,3,5\},\{2,4\}\}$	1,800	3,600	1,200	3,600	600
$\{\{1,4,5\},\{2,3\}\}$	4,200	1,200	1,200	3,600	600
$\{\{1,5\},\{2,3,4\}\}$	600	4,800	1,200	3,600	600
$\{\{1,3\},\{2,4,5\}\}$	1,200	4,200	1,200	3,600	600
$\{\{1,4\},\{2,3,5\}\}$	3,600	1,800	1,200	3,600	600

In this case function v is not superadditive because for any two non-empty disjoint coalitions S and T the following relations are true:

$$\begin{aligned}
v(S\cup T)-v(S)-v(T) &= -(\beta-\alpha)\sum_{i\in S\cup T}\left(1-\frac{k_i}{k(S)+k(T)}\right)n_i \\
&\quad +(\beta-\alpha)\sum_{i\in S}\left(1-\frac{k_i}{k(S)}\right)n_i+(\beta-\alpha)\sum_{i\in T}\left(1-\frac{k_i}{k(T)}\right)n_i \\
&= -(\beta-\alpha)\sum_{i\in S}\frac{k(T)k_i}{k(S)(k(S)+k(T))}n_i \\
&\quad -(\beta-\alpha)\sum_{i\in T}\frac{k(S)k_i}{k(T)(k(S)+k(T))}n_i < 0.
\end{aligned}$$

Negativeness of the expression means that any two disjoint coalitions with ATMs in the region will never benefit from the consolidation of their ATMs in a common network.

Proposition 5. *In a game $(N,v,\overline{\pi})$ with coalition structure $\overline{\pi}$ in case $A=N$, i.e. when each bank from N owns ATMs, there exists a unique stable coalition structure $\overline{\pi}=\{\{1\},\ldots,\{n\}\}$ with respect to the Shapley value where all players are singletons.*

Proof. To prove the existence, note that coalition structure $\overline{\pi}=\{\{1\},\ldots,\{n\}\}$ is stable with respect to the Shapley value because any player i from set N has zero Shapley value component, and when the player deviates from the structure $\overline{\pi}$, the Shapley value component becomes negative following expression (13) (deviation means joining some other player from coalition $N\setminus\{i\}$).

Now prove the uniqueness of stable coalition structure $\overline{\pi}$. Consider any coalition structure $\tilde{\pi}=\{B_1,\ldots,B_m\}\neq\overline{\pi}$. It is obvious that there exists a coalition in coalition structure $\tilde{\pi}$, which contains more than one player from set N. Without loss of generality suppose that this coalition is B_1. Obviously, the Shapley value is not an allocation in game $(N,v,\tilde{\pi})$ because the individual rationality condition is not satisfied that follows from condition $v(B_1)<0$ and equation (13). Therefore, there exists at least one player $j\in B_1$ for which the inequality $\phi_j^{\tilde{\pi}}<0$ holds. It means player j benefits deviating from coalition structure $\tilde{\pi}$ and becoming a singleton. Then player j can guarantee zero Shapley value component himself. Thus, we proved the uniqueness of stable coalition structure $\overline{\pi}$ because we choose coalition structure $\tilde{\pi}$ at random. □

Example 4. Consider a five-player game and suppose that all banks have ATMs in the region, i. e. $A = \{1,2,3,4,5\}$. The number of ATMs that banks own are $k_1 = 10$, $k_2 = 5$, $k_3 = 2$, $k_4 = 4$, $k_5 = 1$. The parameters are: costs per a client's service $\alpha = 0.3$, $\beta = 0.8$, $\gamma = 2$, the number of bank transactions are $n_1 = 10,000$, $n_2 = 7,000$, $n_3 = 2,000$, $n_4 = 6,000$, $n_5 = 1,000$.

Following Proposition 5 there is a unique stable coalition structure with respect to the Shapley value $\overline{\pi} = \{\{1\},\{2\},\{3\},\{4\},\{5\}\}$. The components of the Shapley value are: $\phi_i^{\overline{\pi}} = 0$, $i = 1,\ldots,5$.

4.5 No banks have ATMs: $N \cap A = \emptyset$

Proposition 6. *In a game (N,v,π) where $|N| > 1$ all coalition structures are stable with respect to the Shapley value if and only if $A = \emptyset$, i.e. none bank owns ATMs.*

Proof. *Necessity*. Let all coalition structures in the game (N,v,π) be stable with respect to the Shapley value. Prove that $A = \emptyset$ using method ex adverso. Assume that there exists a player $j \in A$. To prove the statement it is sufficient to consider the coalition structure $\pi = \{\{1\},\ldots,\{n\}\}$. The structure is not stable with respect to the Shapley value since player j may increase his payoff joining any player $i \in N \setminus A$ (see (6)). This contradiction proves the necessity.
Sufficiency. Let $A = \emptyset$, i.e. there is no player in N that owns ATMs. Therefore, $v(S) = 0$ for any coalition $S \subseteq N$. Thus, the Shapley value component for any player and any coalition structure is equal to zero. It means that for all coalition structures the stability condition with respect to the Shapley value is satisfied. □

4.6 Symmetric case

Consider the case when all banks have the same number of clients, $n_i = n_0$, $i \in N$, and all banks with ATMs have the same number of ATMs, $k_i = k_0$ for any $i \in A$.

Consider any coalition S which contains s players, and t players from S have ATMs. The characteristic function can be modified to the following form:

$$v(S) = \begin{cases} n_0(\gamma-\beta)(s-t) - n_0(\beta-\alpha)(t-1), & S \cap A \neq \emptyset, \\ 0, & S \cap A = \emptyset. \end{cases} \tag{14}$$

Now consider a coalition structure π, and a player $i \in S \in \pi$. His component of the Shapley value is calculated by the formula:

$$\phi_i^{\pi} = \begin{cases} \frac{n_0}{s-t}(\gamma-\beta)y, & i \in S \setminus A, \\ \frac{n_0}{t}\left[(\gamma-\beta)(s-t-y) - (\beta-\alpha)(t-1)\right], & i \in S \cap A, \end{cases} \tag{15}$$

where

$$y = \sum_{j=2}^{s} \sum_{\substack{\ell=1 \\ \ell \in [j+t-s,t]}}^{j-1} \frac{\binom{j-1}{\ell}\binom{s-j}{t-\ell}}{\binom{s}{t}}.$$

To obtain formula (15) we use the properties of the Shapley value. According to (14), we conclude players $i, j \in S \cap A$ are symmetrical, and players $p, q \in S \setminus A$ are also symmetrical for any coalition $S \in \pi$. The Shapley value has a property of symmetry, therefore, $\phi_i^\pi = \phi_j^\pi$ and $\phi_p^\pi = \phi_q^\pi$. Following the property of efficiency of the Shapley value

$$v(S) = t\phi_i^\pi + (s-t)\phi_p^\pi,$$

expression (15) for the component of the Shapley value can be easily obtained.

Example 5. Consider again a game with five players. Here we suppose $A = \{1,2,3\}$. The number of ATMs that banks own are $k_0 = 10$, the parameters are: costs per a client's service $\alpha = 0.3$, $\beta = 0.8$, $\gamma = 2$, and the number of bank transactions is $n_0 = 10,000$.

We get 9 out of 52 stable coalition structures with respect to the Shapley value. The corresponding allocations (see expressions (15)) are introduced in Table 4.

Table 4. Stable coalition structures in Example 5

π	ϕ_1^π	ϕ_2^π	ϕ_3^π	ϕ_4^π	ϕ_5^π
$\{\{1\},\{2,3,4,5\}\}$	0	1,500	1,500	8,000	8,000
$\{\{1,4\},\{2,5\},\{3\}\}$	6,000	6,000	0	6,000	6,000
$\{\{1,2,4,5\},\{3\}\}$	1,500	1,500	0	8,000	8,000
$\{\{1,3,4,5\},\{2\}\}$	1,500	0	1,500	8,000	8,000
$\{\{1,5\},\{2\},\{3,4\}\}$	6,000	0	6,000	6,000	6,000
$\{\{1,4\},\{2\},\{3,5\}\}$	6,000	0	6,000	6,000	6,000
$\{\{1,5\},\{2,4\},\{3\}\}$	6,000	6,000	0	6,000	6,000
$\{\{1\},\{2,4\},\{3,5\}\}$	0	6,000	6,000	6,000	6,000
$\{\{1\},\{2,5\},\{3,4\}\}$	0	6,000	6,000	6,000	6,000

4.7 One special case

In this section we investigate a special case: there are banks which do not have their own ATMs, but there are companies that produce and provide an ATM service for the banks. To simplify the model, consider the case when all companies with ATMs and without clients have the same number of ATMs, i.e. $n_i = 0$, $k_i = k_0$ if $i \in A$, and all banks without ATMs have the same number of transactions, i.e. $n_i = n_0$, $k_i = 0$ if $i \in N \setminus A$. It is interesting to find how the profit from cooperation is allocated among the players, and which coalition structures are stable with respect to the Shapley value.

As in the previous subsection, consider a coalition structure π and a player $i \in S \in \pi$. Let $|S| = s$, $|S \cap A| = t$. In this case the characteristic function for $S \subset N$ takes form:

$$v(S) = \begin{cases} n_0(\gamma - \beta)(s-t), & S \cap A \neq \emptyset, \\ 0, & S \cap A = \emptyset. \end{cases} \tag{16}$$

Here we can notice that characteristic function (16) is superadditive. The *i*th component of the Shapley value can be calculated by the following expression:

$$\phi_i^\pi = \begin{cases} n_0(\gamma - \beta)\dfrac{s-t}{t(t+1)}, & i \in S \cap A, \\ n_0(\gamma - \beta)\dfrac{t}{t+1}, & i \in S \setminus A. \end{cases} \tag{17}$$

Formula (17) can be obtained using the expression for characteristic function (16) and the following expression of player *i*'s marginal contribution to a coalition *S* if player *i* owns ATMs:

$$v(S) - v(S \setminus \{i\}) = \begin{cases} n_0(\gamma - \beta)(s-1), & S \cap A = \{i\}, \\ 0, & \text{otherwise}, \end{cases}$$

and if player *i* does not own ATMs:

$$v(S) - v(S \setminus \{i\}) = \begin{cases} n_0(\gamma - \beta), & S \cap A \neq \emptyset, \\ 0, & S \cap A = \emptyset. \end{cases}$$

A player without clients and with ATMs prefers to join a coalition with a larger number of players without ATMs and a less number of players with ATMs, because function $n_0(\gamma - \beta)(s-t)/(t(t+1))$ is a decreasing function of t and an increasing function of $s-t$. A bank with clients and without ATMs prefers to join a coalition with a larger number of players from the set A, because function $n_0(\gamma - \beta)t/(t+1)$ is an increasing function of t and it does not depend on the total number of players from coalition S. The following proposition shows the existence and the configuration of stable coalition structures in the game (N, v, π) with characteristic function (16).

Proposition 7. *In a game (N, v, π) with v given by* (16) *there exists a stable coalition structure with respect to the Shapley value. Moreover, coalition structure $\pi = \{B_1, \ldots, B_m\}$, $m \geq 1$, is stable, iff $|B_j \cap A| = t$ for any $j = 1, \ldots, m$, the following condition holds:*

$$b_{min} \geq \frac{t}{t+2}(b_{max} + 2), \tag{18}$$

where $b_{min} = \min\limits_{j=1,\ldots,m} b_j$, $b_{max} = \max\limits_{j=1,\ldots,m} b_j$, and $b_j = |B_j|$.

Proof. The grand coalition is always stable with respect to the Shapley value because its components are non-negative according to (17), and if a player deviates from the grand coalition, he gets zero payoff as an individual player. Consider a coalition structure $\pi = \{B_1, \ldots, B_m\}$, $m \geq 1$. As we said before, a bank $i \in B_j \in \pi$ with clients and without ATMs prefers coalition with a larger number of players with ATMs. Therefore, player i does not deviate from his current coalition iff all coalitions from coalition structure π have the same number of players with ATMs, i.e. $|B_j \cap A| = t$ for any

$j = 1, \ldots, m$.

Now consider a player $p \in B_j \cap A$ with ATMs and without clients. His payoff is $\phi_p^{\pi} = n_0(\gamma - \beta)(b_j - t)/(t(t+1))$. He can gain only by deviating to a coalition with a larger number of banks with clients. If player p joins a coalition $B_q \in \pi$, he changes coalition structure to π', then his payoff according to the Shapley value (17) becomes $\phi_p^{\pi'} = n_0(\gamma - \beta)(b_q - t)/((t+1)(t+2))$. The condition that player p will not gain by deviating to B_q is the inequality: $\phi_p^{\pi} \geq \phi_p^{\pi'}$, which is equivalent to the following:

$$b_j \geq \frac{t}{t+2}(b_q + 2).$$

This condition must be satisfied for any coalitions B_j and B_q, $b_j \leq b_q$. That is why, we can check this condition for coalitions B_{min} such that $|B_{min}| = b_{min} = \min_j b_j$ and B_{max} such that $|B_{max}| = b_{max} = \max_j b_j$. Therefore, any coalition $S \in \pi$ is stable with respect to the Shapley value against deviation of any player with ATMs and without clients iff the inequality (18) holds. □

Example 6. Consider a five-player game. Let $A = \{1,2,3\}$. The number of ATMs that banks own are $k_1 = k_2 = k_3 = k_0 = 10$, $i = 1,2,3$, $k_4 = k_5 = 0$. The parameters are: costs per a client's service $\alpha = 0.3$, $\beta = 0.8$, $\gamma = 2$, and the number of bank transactions are: $n_1 = n_2 = n_3 = 0$, $n_4 = n_5 = n_0 = 10,000$.

Using (18), we get the only coalition structure $\overline{\pi} = \{\{1,2,3,4,5\}\}$ which is stable with respect to the Shapley value, and the corresponding allocation, calculated by (17), is as follows:

$$\begin{aligned} \phi_1^{\overline{\pi}} = \phi_2^{\overline{\pi}} = \phi_3^{\overline{\pi}} &= 2,000, \\ \phi_4^{\overline{\pi}} = \phi_5^{\overline{\pi}} &= 9,000. \end{aligned}$$

5. Conclusion

We considered the model of a cost reduction problem assuming that banks may cooperate to provide ATM service. In our problem statement, not all possible coalitions are profitable, that is why the theory of games with coalition structures can be used. The question of coalition structure stability is considered in our paper. In specific cases, the existence and the configuration of stable coalition structures with respect to the Shapley value are provided.

Characteristic function (2) does not take into account the quality of the ATM network. In some cases, a large ATM network may be more preferable for clients than a smaller one. This assumption will result in another form of the characteristic function. The developments of the current model are left for future works.

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References

Aumann, R. J. and Drèze, J. H. (1974). Cooperative Games with Coalition Structures. *International Journal of Game Theory*, 3, 217–237.

Bjorndal, E., Hamers, H. and Koster, M. (2004). Cost Allocation in a Bank ATM Network. *Mathematical Methods of Operations Research*, 59, 405–418.

Bloch, F. (1996). Sequential Formation of Coalitions in Games with Externalities and Fixed Payoff Division. *Games and Economic Behavior*, 14, 90–123.

Bogomolnaia, A. and Jackson, M. O. (2002). The Stability of Hedonic Coalition Structures. *Games and Economic Behavior*, 38(2), 201–230.

Driessen, T. and Funaki, Y. (1991). Coincidence of and Collinearity between Game Theoretic Solutions. *OR Spectrum*, 13(1), 15–30.

Faigle, U. and Kern, W. (1992). The Shapley Value for Cooperative Games under Precedence Constrains. *International Journal of Game Theory*, 21, 249–266.

Gow, S. and Thomas, L. (1998). Interchange Fees for Bank ATM Networks. *Naval Research Logistics*, 45, 407–417.

Haeringer, G. (2001). Stable Coalition Structures with Fixed Decision Scheme. Economics with Heterogeneous Interacting Agents. *Lecture Notes in Economics and Mathematical Systems*, 503(IV), 217–230.

Hart, S. and Kurz, M. (1983). Endogenous Formation of Coalitions. *Econometrica*, 52, 1047–1064.

Katsev, I. and Yanovskaya, E. (2013). The Prenucleolus for Games with Restricted Cooperation. *Mathematical Social Sciences*, 66, 56–65.

Marini, M. A. (2009). Games of Coalition and Network Formation: A Survey. Networks, Topology and Dynamics. *Lecture Notes in Economics and Mathematical Systems*, 613(II), 67–93.

Naumova, N. (2012). Generalized Proportional Solutions to Games with Restricted Cooperation. In Petrosyan, L. A., Zenkevich, N. (eds.), *Contributions to Game Theory and Management*. Saint Petersburg, Graguate School of Management SPbU, 5, 230–242.

Nouweland, A., Borm, P., Golstein Brouwers, W., Groot Bruinderink, R. and Tijs, S. (1996). A Game Theoretic Approach to Problems in Telecommunications. *Management Sciences*, 42(2), 294–303.

Parilina, E. (2007). A Game-Theoretic Approach to Resource Sharing Management. *Proceedings of The Second International Conference on Game Theory and Applications*. World Academic Press, 49–52.

Petrosjan, L. and Mamkina, S. (2006). Dynamic Games with Coalitional Structures. *International Game Theory Review*, 8(2), 295–307.

Petrosyan, L. A, Sedakov, A. A. and Syurin, A. N. (2006). Multistage Games with Coalitional Structure. *Vestnik of St. Petersburg University*, 10(4), 97–110 (in Russian).

Schmeidler, D. (1969). The Nucleolus of a Characteristic Function Game. *SIAM Journal of Applied Mathematics*, 17, 1163–1170.

Sedakov, A., Parilina, E., Volobuev, Y. and Klimuk, D. (2013). Existence of Stable Coalition Structures in Three-Person Games. In Petrosyan, L. A., Zenkevich, N. (eds.), *Contributions to Game Theory and Management*. Saint Petersburg, Graguate School of Management SPbU, 6, 407–422.

Shapley, L. S. (1953). A Value for *n*-Person Games. In Kuhn, H. W., Tucker, A. W. (eds.), *Contributions to the Theory of Games*. Princeton, Princeton University Press, II, 307–317.

Tiebout, C. (1956). A Pure Theory of Local Public Expenditures. *Journal of Political Economy*, 65, 319–337.