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Banzhaf Like Value for Games with Interval Uncertainty

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Abstract This paper focuses on the Banzhaf value for cooperative games with a finite set of players where the coalition values, expressed by the characteristic function, are compact intervals of the real numbers. We generalize the Banzhaf value for TU-cooperative games to the class of games with interval uncertainty which have many applications. Furthermore the Banzhaf like value is here characterized through some axioms.

Keywords Cooperative situations, interval values games, Banzhaf index
JEL classification C71

1. Introduction

Since some years game theory is more and more oriented to applications of economics, operation research, engineering and being game theory a mathematical discipline, *games* are interesting and new mathematical objects worth of investigation. In the last years many applications were studied about *games with interval uncertainty*. Uncertainty on coalition values led to new models of cooperative games and to corresponding solutions. In general Interval Analysis is a framework for calculations with intervals and analyzing uncertainty models. It is successfully applied also in global optimization (see Hansen 2009; Ratschek and Voller 1991). The study of interval uncertainty was introduced in Moore (1979) and inspired the paper Yager and Kreinovich (2000). About this new class of games see also Alparslan-Gök et al. (2010) and Mallozzi et al. (2011).

In this paper we focus our attention on Banzhaf value as measure of power in decision making. Considering the interpretation of power as the chance to be critical for a decision, Banzhaf proposed his index criticizing the weights for the coalitions made by Shapley-Shubik (see Gonzales-Diaz et al. 2010, Peters 2008). Many authors propose axiomatizations about power indices (see Owen 1978; Nowak 1997; Albizuri 2001; Khmelnitskaya 1999).

In Laruelle et al. (2001) a new interesting axiomatization of Banzhaf index was given as measures of power in strategic procedures in some subclasses of TU-games as simple games or simple superadditive ones. By TU-games we mean the cooperative games with transferable utility (see Peters 2008 for more details).

We have to underline that the interval games models the situations where the decision makers wish to cooperate and they know with certainty only lower and upper

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bound of revenues and costs.

In Carpente et al. (2008) coalitional interval games were associated to strategic interval games. Furthermore reward/cost sharing models can be studied successfully through games with interval uncertainty.

We can note that any interval of real numbers can be considered as a vector in $\mathbb{R}^2$, so an interval game can be studied as a particular vector valued game (taking into account the hypotheses made on the intervals and the operations on them). At this light interval cooperative games can be seen as a class of partially ordered games (see Puerto et al. 2008).

A remark has to be done: we do not consider probabilistic hypotheses about the uncertainty payoff, these games (dealing with stochastic payoffs) were studied in Suijs et al. (1999).

The outline of the paper is the following: Section 2 recalls basic concepts, definitions about cooperative games and the algebra of interval analysis, Section 3 contains results about Banzhaf value for games with intervals uncertainty; Section 4 deals with the axiomatization and properties of Banzhaf like value. In Section 5 we write the conclusions and some ideas for open problems. The paper ends with a large number of references cited in the text and useful to interested readers.

2. Background

We will write $\Im(\mathbb{R})$ for the set of non-empty compact intervals of $\mathbb{R}$. Here we write some operations in the algebra of intervals. They are supposed to be non-empty and compact intervals of real numbers.

Let $a,b,c,d \in \mathbb{R}$, $\alpha \in [0,+\infty)$ with $a \leq b$; $c \leq d$. Let us put:

(i) $[a,b] = [c,d] \Leftrightarrow a = c$ and $b = d$;

(ii) $[a,b]+[c,d] = [a+c,b+d]$;

(iii) $[a,b]-[c,d] = [a,b]+[-d,-c] = [a-d,b-c]$;

(iv) $\alpha[a,b] = [\alpha a, \alpha b]$;

(v) $[a,b] \geq [c,d] \Leftrightarrow a \geq c$ and $b \geq d$;

(vi) $[a,b] > [c,d] \Leftrightarrow a > c$ and $b > d$ (for further details see Moore 1979).

A cooperative n-person TU-game in coalitional form and with interval uncertainty, is an ordered pair $< N,w >$ where $N = \{1,2,\ldots,n\}$ is the set of players and w is the characteristic function, $w : 2^N \to \Im(\mathbb{R})$ which assigns to each nonempty coalition S, an interval of real numbers $w(S) \in \Im(\mathbb{R})$, $w(S)$ is called the worth of coalition S and it falls in an uncertainty interval $[\underline{w}(S), \overline{w}(S)]$, $w(\emptyset) = [0,0]$. We denote the infimum of the interval $\underline{w}(S) = \inf w(S)$, the supremum of the interval $\overline{w}(S) = \sup w(S)$.

We call $\Im G^N$ the set of interval TU-games with N players. Given an interval TU-game $\langle N,w \rangle$ we call two players i,j *symmetric players* if $w(S \cup i) = w(S \cup j)$ $\forall S \subset N \setminus \{i,j\}$. We call i a *dummy player* if $w(S \cup i) - w(S) = w(\{i\})$, $\forall S \subset 2^{N \setminus \{i\}}$. We call i a *null player* if $w(S \cup i) = w(S)$, $\forall S \subset 2^{N \setminus \{i\}}$.

Definition 1. *A simple interval game is a game where every coalition has value the interval* $[0,0]$ *or* $[1,1]$*, and* $w(N) = [1,1]$*. Coalitions with value the interval* $[1,1]$ *are called winning and coalitions with value* $[0,0]$ *are called losing.*

Remark 1. Note that the algebra on intervals is not the usual algebra on $\mathbb{R}$. For example: $[1,3]+[-5,-2] = [-4,1]$ and $[1,3]+[-5,-2] \neq [1,3]-[5,2]$, when the last one has no meaning as an interval.

It is natural to define the "subtraction" as in Moore (1979), in fact if $a \leq x \leq b$, $c \leq y \leq d$, then $a-d \leq x-y \leq b-c$, but in this way some important properties about games (as efficient property) are missed. Let $[a,b],[c,d]$ be two intervals, we define $|[a,b]| = b-a$. So in the whole paper we restrict ourselves to a class of intervals where $|[a,b]| \geq |[c,d]|$ and the subtraction is defined as follows: $[a,b]-[c,d]$= $[a-c,b-d]$ so we can introduce the classes of interval games SMIG^N and KIG^N as made in Alparslan-Gök et al. (2010).

By SMIG^N we denote the class of size monotonic interval games with N players, by KIG^N we denote the class of interval games generated by the cone K as better explained in the following definition. Note that $[a-c,b-d] \subset [a-d,b-c]$.

The TU-games with interval uncertainty (which we will call briefly "interval games") are a generalization of the usual TU-cooperative games.

Definition 2. *We call a game* $\langle N,w\rangle$ *size monotonic game if* $\langle N,|w|\rangle := \langle N,\overline{w}-\underline{w}\rangle$ *is monotonic, i.e.*

$$|w|(S) \leq |w|(T) \quad \forall S,T \in 2^N, S \subset T.$$

Intuitively if $S \subset T$*, the measure of the interval given by coalition* S *is less or equal to the measure of the interval given by the coalition* T*.*

We note two facts:

(i) if w is a *size monotonic game* then $w(S\cup\{i\})-w(S)$ (which is called interval marginal vector) has meaning for all $S \in N$, $i \notin S$;

(ii) the interval marginal vectors of a size monotonic game are efficient, that is $\sum_{i=1}^n w(S\cup\{i\})-w(S) = w(N)$.

Let us consider the unanimity game u_T of the classical theory defined in the following:

$$u_T(S) = \begin{cases} 1 & \text{if } T \subset S,\ S,T \subset 2^N \setminus \{\emptyset\} \\ 0 & \text{otherwise} \end{cases}$$

and, given $I \in \Im(\mathbb{R})$, define the unanimity like game as follows:

$$Iu_T(S) = u_T(S)I = \begin{cases} I & \text{if } T \subset S \\ [0,0] & \text{otherwise} \end{cases}$$

By KIG^N we mean the additive cone generated by the set

$$K = \{I_T u_T : T \in 2^N \setminus \{\emptyset\}, I_T \in \Im(\mathbb{R})\}.$$

By I_T we mean a real interval depending on the coalition T. In this way, each element of the cone is a finite sum of elements of K, in other words if $w \in \mathrm{KIG}^N$, then $w = \sum_{T \in 2^N \setminus \{\emptyset\}} I_T u_T$. It is easy to prove that $\mathrm{KIG}^N \subset \mathrm{SMIG}^N$.

3. Banzhaf interval value

Let us define the *Banzhaf like index* for a generic interval game $\langle N, w\rangle$. Let

$$\theta_i(w) = \sum_{S \subset N, i \notin S} \{w(S \cup i) - w(S)\}.$$

Let us define $B_i : \Im G^N \to \Im(\mathbb{R})$ the *Banzhaf like index* for the player i in the following way:

$$\begin{aligned} B_i(w) &= \frac{\theta_i(w)}{2^{|N|-1}} = \sum_{S \subset N, i \notin S} \frac{\{w(S \cup i) - w(S)\}}{2^{|N|-1}} = \\ &= \sum_{S \subset N, i \notin S} \frac{\{[\underline{w}(S \cup i), \overline{w}(S \cup i)] - [\underline{w}(S), \overline{w}(S)]\}}{2^{|N|-1}}. \end{aligned}$$

The Banzhaf like index will be denoted $B = (B_1, \ldots, B_N)$.

Example 1. Let G be the cooperative game in the table below:

S	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
$w(S)$	[2,3]	[3,4]	[4,5]	[6,8]	[6,8]	[9,11]	[12,15]

By this notation we mean $w(\{1\}) = [2,3]$, $w(\{2,3\}) = [9,11]$ and so on. The Banzhaf like value is: $B(w) = \big(B_1(w), B_2(w), B_3(w)\big) = ([2, 11/4], [15/4, 9/2], [4, 19/4])$. In fact, $B_1(w) = \{w(1,2) - w(2) + w(1,3) - w(3) + w(1,2,3) - w(2,3)\} = [6,8] - [3,4] + [6,8] - [4,5] + [12,15] - [9,11] = [8,11]/4$. The same for B_2 and B_3.

Proposition 1. $B(w) = [B(\underline{w}), B(\overline{w})]$ $\forall w \in \mathit{SMIG}^N$

Proof.

$$\begin{aligned} B_i(w) &= \sum_{S \subset N, i \notin S} \frac{\{[\underline{w}(S \cup i) - \underline{w}(S), \overline{w}(S \cup i) - \overline{w}(S)]\}}{2^{|N|-1}} = \\ &= \left[\sum_{S \subset N, i \notin S} \frac{\{\underline{w}(S \cup i) - \underline{w}(S)\}}{2^{|N|-1}}, \sum_{S \subset N, i \notin S} \frac{\{\overline{w}(S \cup i) - \overline{w}(S)\}}{2^{|N|-1}}\right] = \\ &= [B_i(\underline{w}), B_i(\overline{w})]. \qquad \square \end{aligned}$$

In this paper we wish to give an axiomatization of the interval Banzhaf value on the line of Nowak (1997) given for scalar games.

Let us define a *reduced game*. Let $\langle N, w\rangle$ be an interval game with at least two players i, j; we call $p = \{i, j\}$, $M = N \setminus p$. We define the interval game $\langle M \cup \{p\}, w_p\rangle$ in the following way: $w_p(S) = w(S)$ and $w_p(S \cup \{p\}) = w(S \cup p)$, $\forall S \subseteq M$ in the meaning that $\forall S \subseteq M$ it turns out: $[\underline{w}_p(S), \overline{w}_p(S)] = [\underline{w}(S), \overline{w}(S)]$ and $[\underline{w}_p(S \cup \{p\}), \overline{w}_p(S \cup \{p\})] = [\underline{w}(S \cup p), \overline{w}(S \cup p)]$. Clearly w_p is the characteristic function of an interval game with

$n-1$ players obtained by amalgamating the players i and j of the game $\langle N,w\rangle$ into one player called p.

Let us consider the following axioms: 2-EFF, SYM, DUM, SMON. We say that a solution $\psi : \Im G^N \to \Im(\mathbb{R})^n$ verifies:

- 2-EFF (2-efficiency property): $\psi_i(w)+\psi_j(w)=\psi_p(w)$ $\forall w,i,j,p$ as above.
- SYM (symmetry): $\psi_i(w)=\psi_j(w)$ $\forall i,j$ such that $w(S\cup i)=w(S\cup j)$ $\forall S\subset N\setminus\{i,j\}$ (i and j symmetric players).
- DUM (dummy property): $\psi_i(w)=w(\{i\})$ $\forall i$ dummy player.
- SMON (strong monotonicity): for all $w,z\in\Im G^N$, if $w(S\cup i)-w(S)\geq z(S\cup i)-z(S)$ then $\psi(w)\geq\psi(z)$.

Theorem 1. *The interval Banzhaff value verifies 2-EFF, SYM, DUM, SMON axioms on SMIG^N.*

Proof.

- 2-EFF:

$$\begin{aligned}
B_p(w_p) &= \textstyle\sum_{S\subset(N\setminus p)\cup\{p\},p\notin S}\frac{1}{2^{|N|-2}}\{w_p(S\cup p)-w_p(S)\} = \\
&= \textstyle\sum_{S\subset(N\setminus\{i,j\}),\{i,j\}\notin S}\frac{1}{2^{|N|-2}}\{w(S\cup\{i,j\})-w(S)\} = \\
&= \textstyle\sum_{S\subset(N\setminus\{i,j\}),\{i,j\}\notin S}\frac{1}{2^{|N|-1}}\{2w(S\cup\{i,j\})-2w(S)\}.
\end{aligned}$$

  It turns out:

$$\begin{aligned}
\{2w(S\cup\{i,j\})-2w(S)\} &= \{w(S\cup\{i,j\})-w(S\cup i)+w(S\cup j)-w(S)\}+ \\
&\quad +\{w(S\cup\{i,j\})-w(S\cup j)+w(S\cup i)-w(S)\}.
\end{aligned}$$

  Then,

$$\begin{aligned}
&\textstyle\sum_{S\subset(N\setminus\{i,j\}),\{i,j\}\notin S}\frac{1}{2^{|N|-1}}\{w(S\cup\{i,j\})-w(S\cup i)+w(S\cup j)-w(S)\}+ \\
&+\textstyle\sum_{S\subset(N\setminus\{i,j\}),\{i,j\}\notin S}\frac{1}{2^{|N|-1}}\{w(S\cup\{i,j\})-w(S\cup j)+w(S\cup i)-w(S)\} = \\
&=\textstyle\sum_{S\subset(N\setminus j),j\notin S}\frac{1}{2^{|N|-1}}\{w(S\cup j)-w(S)\}+\sum_{S\subset(N\setminus i),i\notin S}\frac{1}{2^{|N|-1}}\{w(S\cup i)-w(S)\} \\
&=B_i(w)+B_j(w).
\end{aligned}$$

  So $B_p(w_p)=B_i(w)+B_j(w)$ and this proves the first axiom. Remark that in the proof we are able to do $\{w_p(S\cup\{p\})-w_p(S)\}$ because $w\in SMIG^N$ and so w_p. In general it is not possible adding and subtracting addends but this is possible in the class SMIG^N.

- SYM: that is $B_i(w)=B_j(w)$ $\forall i,j$ symmetric players. This follows from the definition, in fact $B_i(w)=[B_i(\underline{w}),B_i(\overline{w})]$, $B_j(w)=[B_j(\underline{w}),B_j(\overline{w})]$ $\forall w\in$ SMIG^N.

- DUM:
$$B_i(w) = w(\{i\}) \text{ if } w(S\cup i) - w(S) = w(\{i\}).$$

- SMON:
$w(S\cup i)-w(S) \geq z(S\cup i)-z(S)$, $w,z \in \text{SMIG}^N$ means that $[\underline{w}(S\cup i)-\underline{w}(S), \overline{w}(S\cup i)-\overline{w}(S)] \geq [\underline{z}(S\cup i)-\underline{z}(S), \overline{z}(S\cup i)-\overline{z}(S)]$ and this completes the proof of the axioms. □

Theorem 2. *An interval value ψ verfies 2-EFF, SYM, DUM, SMON axioms if and only if ψ is the Banzhaf like value on the game sets KIG^N.*

Proof. We have already seen (by Theorem 1) that the Banzhaf like value verifies the said axioms on SMIG^N. For the converse, let ψ be a value verifying the four axioms, let us prove that $\psi = B$ on KIG^N.

We know from the definition of Banzhaf like value that:

$$B_i(Iu_T) = B_i(u_T(S)I) = \begin{cases} \dfrac{I}{2^{|T|-1}} & \text{if} \quad i\in T \\ [0,0] & \text{if} \quad i \notin T \end{cases} \tag{1}$$

(i) Let us consider the unanimity like game as introduced in Section 2 and we first prove that $\psi_i(Iu_T)$ verifies the (1). If $|T| = 1$ each player is dummy, so

$$\psi_i(Iu_T) = \psi(u_T(i)I) = \begin{cases} I & \text{if} \quad i\in T \\ [0,0] & \text{otherwise} \end{cases}$$

So (1) is true for ψ.

Now, by induction, let us suppose that (1) is true for coalition T such that $|T| \leq k$ or $|N| \leq m$ and let us consider an unanimity like game Iu_T with $m+1$ players and T has $k+1$ players.

Let $i, j \in T$, call $p = \{i, j\}$ and let us study the game $(Iu_T)_p$. Then $(Iu_T)_p$ is the unanimity like game with m players and with coalition $T' = (T\setminus p)\cup\{p\})$ and $|T'| = k$. By induction hypothesis

$$\psi_p((Iu_T)_p) = \frac{I}{2^{|T'|-1}} = \frac{I}{2^{k-1}},$$

by 2-EFF it follows

$$\psi_i(Iu_T) + \psi_j(Iu_T) = \psi_p((Iu_T)_p) = \frac{I}{2^k-1}.$$

By SYM it follows $\psi_i(w) = \psi_j(w)$ $\forall i, j$ symmetric players. So it follows that $\psi_i(Iu_T) = \frac{I}{2^k}$=$\frac{I}{2^{|T|-1}}$ and by DUM it follows that $\psi_j(Iu_T) = [0,0]$ if $j \notin T$. Then ψ is the Banzhaf like value on unanimity like game for any finite set of players.

(ii) Let us consider a generic interval game $\langle N, w\rangle$, $w \in \text{KIG}^N$, $w = \sum_{T\in 2^N\setminus\{\emptyset\}} I_T u_T$ and denote by $\eta(w)$ the number of non null coefficients in this representation.

We will use the induction principle on the number $\eta(w)$ and on the number of players to complete the proof. If $\eta(w) = 1$ for the previous step about unanimity like game, it turns out $\psi(w) = B(w)$ (independently on the number of players).

Let us suppose that $\psi(w) = B(w)$ on any interval game w with at most n players and for any game (call it again w) s.t. $\eta(w) \leq k$, $k \in \mathbb{N}$ and $n+1$ players. Let w be an interval game with $n+1$ players and $\eta(w) = k+1$. So there are $k+1$ different coalitions $T_1, \ldots, T_{k+1} \neq \emptyset$ and such that

$$w = \sum_{r=1}^{k+1} I_{T_r} u_{T_r},$$

where all the coefficients are not the null intervals.

Let $T = T_1 \cap T_2 \ldots \cap T_{k+1}$ and being $k+1 \geq 2$ it turns out $N \setminus T \neq \emptyset$. If we suppose $i \notin T$ we can define the game:

$$\tilde{w} = \sum_{r: i \in T_r} I_{T_r} u_{T_r},$$

the $\eta(\tilde{w}) \leq k$ and $w(S \cap i) - w(S) = \tilde{w}(S \cap i) - \tilde{w}(S)$ $\forall S$ s.t. $i \notin S$.
From this relation and the SMON axiom we can write that $\psi_i(w) = \psi_i(\tilde{w})$ and from inductive hypothesis $\psi_i(\tilde{w}) = B(\tilde{w})$ $\forall i \in N \setminus T$, so we can conclude that

$$\psi_i(w) = B_i(w) \ \forall i \in N \setminus T. \tag{2}$$

Now we consider $j \in T$ and $i \in N \setminus T$, call $p = \{i, j\}$ and we can consider the reduced game w_p. This game has n players so for the inductive hypothesis it turns out

$$\psi_p(w_p) = B_p(w_p). \tag{3}$$

At this point we remember that we can apply the 2-EFF axiom to ψ and B, so we have:

$$\psi_p(w_p) = \psi_i(w) + \psi_j(w), \ B_p(w_p) = B_i(w) + B_j(w). \tag{4}$$

Taking into account (2), (3) we can write $\psi_j(w) = B_j(w)$ $\forall j \in T$ and with (4) we get the thesis for all $i \in N$. (For more details about this proof see Nowak (1997) in the classical case). □

4. Properties about Banzhaf like value

Given two intervals $[a_1, b_1]$ and $[a_2, b_2]$, we define the maximum and minimum between two intervals in the following way:

$$\max\{[a_1, b_1], [a_2, b_2]\} = [\max a_i, \max b_i], \quad i = 1, 2$$
$$\min\{[a_1, b_1], [a_2, b_2]\} = [\min a_i, \min b_i], \quad i = 1, 2$$

When we write $(w \vee z)$ or $(w \wedge z)$ we mean the maximum and minimum (respectively) between two intervals.

Definition 3. *Let $w,z \in \Im G^N$. The maximum game of w,z, is defined for each $S \subset N$ by $w \vee z = max\{w(S), z(S)\}$. In a similar way the minimum game of w,z, is defined for each $S \subset N$ by $w \wedge z = min\{w(S), z(S)\}$.*

We say that an allocation rule ϕ verifies:

- TF (transfer) property:
  if $\forall w,z \in \Im S^N$ (simple interval game) $\phi(w \vee z) + \phi(w \wedge z) = \phi(w) + \phi(z)$;
- EFF (efficient) property: if $\forall w \in \Im G^N$, $\sum_N \phi_i(w) = w(N)$;
- NPP (null player) property:
  if $\forall w \in \Im G^N$, $\forall i, j \in N$; symmetric players, it turns out $\phi_i(w) = \phi_j(w)$;
- ADD (additional) property:
  if $\forall w,z \in \Im G^N$ it turns out $\phi(w+z) = \phi(w) + \phi(z)$.

Theorem 3. *The Banzhaf like value is an allocation rule which verifies the ADD, NPP, SYM, TF properties.*

Proof. The Banzhaf like value verifies the NPP property in fact if i, j are symmetric players that is $w(S \cup i) = w(S \cup j)$ then $B_i(w) = B_j(w)$. The SYM property has been already proved.

Let us prove TF property. Let $v, w \in \Im S^N$.

$$\begin{aligned} B_i(w) + B_i(v) &= \sum_{S \subset (N \setminus i), i \notin S} \frac{1}{2^{|N|-1}} \{w(S \cup i) - w(S)\} + \\ &\quad + \sum_{S \subset (N \setminus i), i \notin S} \frac{1}{2^{|N|-1}} \{v(S \cup i) - v(S)\} = \\ &= B_i[\underline{(w+v)}, \overline{(w+v)}] = B_i(w+v) \end{aligned}$$

That is the Banzahf like value verifies the ADD property.

$$B_i(w \vee v) = B_i[\max\{\underline{w}, \underline{v}\}, \max\{\overline{w}, \overline{v}\}] = \begin{cases} B_i[\underline{w}, \overline{w}] & \text{if} \quad \underline{w} \geq \underline{v}; \overline{w} \geq \overline{v} \\ B_i[\underline{w}, \overline{v}] & \text{if} \quad \underline{w} \geq \underline{v}; \overline{w} \leq \overline{v} \\ B_i[\underline{v}, \overline{v}] & \text{if} \quad \underline{v} \geq \underline{w}; \overline{v} \geq \overline{w} \\ B_i[\underline{v}, \overline{w}] & \text{if} \quad \underline{v} \geq \underline{w}; \overline{v} \leq \overline{w} \end{cases}$$

Analogously:

$$B_i(w \wedge v) = B_i[\min\{\underline{w}, \underline{v}\}, \min\{\overline{w}, \overline{v}\}] = \begin{cases} B_i[\underline{v}, \overline{v}] & \text{if} \quad \underline{w} \geq \underline{v}; \overline{w} \geq \overline{v} \\ B_i[\underline{v}, \overline{w}] & \text{if} \quad \underline{w} \geq \underline{v}; \overline{w} \leq \overline{v} \\ B_i[\underline{w}, \overline{w}] & \text{if} \quad \underline{v} \geq \underline{w}; \overline{v} \geq \overline{w} \\ B_i[\underline{w}, \overline{v}] & \text{if} \quad \underline{v} \geq \underline{w}; \; \overline{v} \leq \overline{w} \end{cases}$$

So we obtain:

$$B_i(w \vee v) + B_i(w \wedge v) = \begin{cases} B_i(w) + B_i(v) & \text{if} \quad \underline{w} \geq \underline{v}; \overline{w} \geq \overline{v} \\ B_i[\underline{w}, \overline{v}] + B_i[\underline{v}, \overline{w}] & \text{if} \quad \underline{w} \geq \underline{v}; \overline{w} \leq \overline{v} \\ B_i(v) + B_i(w) & \text{if} \quad \underline{v} \geq \underline{w}; \overline{v} \geq \overline{w} \\ B_i[\underline{v}, \overline{w}] + B_i[\underline{w}, \overline{v}] & \text{if} \quad \underline{v} \geq \underline{w}; \overline{v} \leq \overline{w} \end{cases}$$

We have to study the second and the fourth cases. Let us consider the second case (the fourth is analogous). We can think of two new interval games z, h such that $\forall S \subset 2^N \setminus \{\emptyset\}$ $z(S) = [\underline{w}(S), \overline{v}(S)]$ and $h(S) = [\underline{v}(S), \overline{w}(S)]$.

$$\begin{aligned} B_i(w \vee v) + B_i(w \wedge v) &= B_i[\underline{w}, \overline{v}] + B_i[\underline{v}, \overline{w}] = B_i(z) + B_i(h) = \\ &= [B_i(\underline{z}), B_i(\overline{z})] + [B_i(\underline{h}), B_i(\overline{h})] = \\ &= [B_i(\underline{w}), B_i(\overline{v})] + [B_i(\underline{v}), B_i(\overline{w})] = \\ &= [B_i(\underline{w}) + B_i(\underline{v}), B_i(\overline{v}) + B_i(\overline{w})] = B_i(v+w). \quad \square \end{aligned}$$

5. Conclusion and open problems

In this paper we have studied a Banzhaf like value for games with interval uncertainty. We have considered some properties (symmetry, 2-efficiency, dummy, strong monotonicity) which characterizes the power index mentioned and we have generalized them to this new challenge class of games.

Some open problems are the following:

(i) other axiomatizations of Banzhaf like power index can be given;

(ii) starting from the idea that in the applications of reality, decision makers have not one but many criteria "to optimize", keeping into account Pusillo and Tijs (2012), we can investigate which properties are valid for multicriteria TU-games (or vector TU-games);

(iii) we can study under which hypotheses a non cooperative game can be trasformed into a cooperative one. So we can define the Banzhaf like value for a non cooperative game as the Banzhaf like value for the corresponding cooperative game and to study its properties.

Some of these issues are works in progress.

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