AUCO Czech Economic Review 5 (2011) 306–323
Acta Universitatis Carolinae Oeconomica

Optimizing the Efficiency of Weighted Voting Games

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Received 20 December 2010; Accepted 7 October 2011

Abstract Having a group of voters endowed with weights, the simple weighted voting game (or system) represents a system of approving propositions in which the approved is only a proposition that is accepted by voters weighted to a number that is at least equal to a prescribed number called a quota. We call the system simple if there is only one set of weights and one quota, as opposed to the multi-rule systems that have more weights assigned to each voter and with more quotas. This paper presents an analysis of the efficiency of simple weighted voting systems. It assumes the Impartial Anonymous Culture (the probability of a single voter voting for a proposition is $1/2$ and voters act independently). This culture is used for the general evaluation of voting systems, when no specific information about propositions and voters' preferences are known, or when the voters' preferences and proposition characteristics are not willing to be reflected in the voting system itself, keeping in mind its non-pragmatics, fairness and generality. The efficiency of a simple weighted voting system is defined as the probability of a proposition being approved. This paper focuses on efficiency maximization and minimization with respect to weights. We prove a theorem which enables the computing of the efficiency maximum and efficiency minimum with respect to weights, given the number of voters and quota in linear time.

Keywords Weighted voting game, integer programming, efficiency of voting, IAC models
JEL classification D71, D72

1. Introduction

Voting systems are systems transforming many individual preferences into one. By simple weighted voting system we mean a voting system that processes a proposition and assigns the proposition a single value: accepted or rejected. If accepted, the proposition is applied and changes the status quo. If rejected, it is not applied and does not change the status quo. The proposition is accepted within the system if, and only if, the sum of weights that are assigned to all the voters who accept the proposition is at least equal to a prescribed real number called the quota. For simplicity we assume that all the weights are real numbers between 0 and 1 (including both) and in addition the weights add up to 1 (each simple weighted voting system can be normalized in this way). Simple weighted voting systems are used in various institutions all around the world, including parliamentary and legislative institutions, United Nations institutions, European institutions, general committees of shareholders, Academy Award committees, the Nobel prize committee, governments at the national, district or city

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level, executive boards of national banks, courts of law, housing associations and many other institutions. Voting plays a crucial role not only in political markets, but also in traditional economic and financial markets, since political decisions largely impact consumption, investment, inflation, unemployment and overall economic performance through fiscal and monetary policies. For the use of weighted voting systems in international organizations, see for example McIntyre (1954).

We consider two types of propositions: The first type we call simple, the second we call complex. Simple propositions are all those that can be inverted to their opposite. This means they specify such a change that admits only one alternative proposition. This alternative contradicts the original one proposed. An example of a simple proposition is one concerned with entering the European Union, as the only alternative to entering is not entering. For simple propositions there are generally only two states of nature. On the other hand, complex propositions have many alternatives. An example of a complex proposition is changing a tax law, as the proposed change usually has many alternatives that cannot be proposed individually. Assume the law to be very simple, stating that the tax for individuals is 20% of their income. Then there is no single proposition on tax law in contradiction with the original. The tax rate can be 25% or 15%. Both these tax rates are possible alternative propositions in contradiction with the 20% tax rate. Hence the proposition on the tax rate is a complex proposition. In other words, simple propositions are those dealing with changes that admit only two states of nature and complex propositions are those which admit at least three.

The reason for distinguishing simple and complex propositions is to avoid objections concerning simple weighted voting systems with quotas under one half, which we admit. Sometimes there are complaints that once a proposition is rejected, its opposite should be accepted under the same voting system. This simple behavior is clearly not assured for quotas below one half. However, these objections are relevant only for simple propositions, since for a complex proposition there is no single opposite proposition and so once a complex proposition is rejected, some of its contradictory propositions might be rejected as well under the same preference setting. Additionally in some cases, only certain types of propositions can be processed under the weighted voting system with a quota lower than one half—for example in a general assembly of stock holders or in bankruptcy proceedings.[1] In many codes of rules over public and private associations or political parties, the most powerful authority (usually a congress, convention or assembly of all members) can be convened by less than half of all the members. Although, to convene an authority may not be considered approving a proposition, it actually is an approval process changing the status quo. An important role the quotas below one half also play in the multi-rule voting systems, where more than just one rule must be fulfilled for a coalition to win. For example, consider a union of 10 countries, where two of them are very large and, based on their population, they would be the majority on their own. But, such a union is hardly acceptable for the other 8 countries, which would have no practical power in the union under the population majority rule. For such a union it might be useful, aside from the population

[1] Under Czech bankruptcy law, a creditor assembly can be convened when supported by at least two creditors with at least a 10% share of the total value of all submitted receivables, i.e. the quota is 0.1.

rule, to apply another rule requiring (for example) at least 4 countries to accept the proposition for it to be approved. The quota would then be 0.4 in this rule and it makes sense to apply it. This reasoning leads us to admit all possible quotas between 0 and 1.

Weights can represent the share of mandates in some voting bodies, the shares of property values of stock holders or creditors, the populations of countries in supranational institutions, and the populations of counties or districts in national institutions.[2]

If we wish to study the voting systems themselves, while not having any information about the preferences of single voters, parties or voting countries, the natural way would be to assign each voter a theoretical probability of accepting any proposition. We call it the ”probability of acceptance”. This probability is considered to be the same for all voters, since we do not have any information about future propositions. Having no such information about the propositions and voters' preferences, it is natural to assume this probability is 0.5 and the voters act independently. Putting these assumptions together is often called an IAC (Impartial Anonymous Culture) assumption.

The probability of a proposition being approved we call the "probability of approval". We will study the probability of approval (which we will also call "an efficiency of the voting system") under the IAC assumption, since it is one of the major characteristics of a voting procedure itself. This efficiency is also known in the literature as the Coleman index or the power of a collectivity to act, and has been introduced in Coleman (1971) and used in Leech (2002), for example. A slight generalization of the Coleman index is studied in Lindner (2008). The Coleman index can be considered an a priori voting power assigned to each voter.

We are interested in the behavior of a simple weighted voting system efficiency as a function of weights and quota. The questions we raise and answer are:

(i) What are the minima and maxima of the efficiency function with respect to the weights and can they be expressed analytically in terms of quota and number of voters?

(ii) Is the efficiency function symmetric around the quota equal to one half?

(iii) Are there any intervals in which the quota can move freely so that the efficiency maximum (resp. efficiency minimum) with respect to the weights does not change? If so, how they can be formally described?

(iv) How many different values can the efficiency maximum, resp. efficiency minimum, attain in terms of the number of voters? Can this number be expressed explicitly as a function of n?

The concept of efficiency is quite clear, but the problem of computing the exact efficiency of a simple weighted voting system is the time complexity. This was shown to be an *NP*-complete problem, see Matsui and Matsui (2000, 2001) or Prasad and Kelly (1990). We do not study multi-rule voting systems—see for example Leech et al. (2007), Leech and Aziz (2008) or Leech (2002)—as it would not bring any further significant insight to our analysis. For further information on weighted voting systems

[2] For example, in the Lower House of the Czech Parliament, the shares of mandates assigned to districts depend upon the number of valid votes cast in each district.

and their efficiency, see Hosli (2008) (balancing legitimacy and efficiency), Zuckerman et al. (2008) (theoretical analysis of individual voting power), Aziz and Paterson (2008) (survey on computationally tractable weighted voting systems) and Felsenthal and Machover (1998). In the article from Felsenthal and Machover (2004), the authors defend the concept of a priori voting power opposing some of the objections being made to this concept by explicitly stating: *"The main purpose of measuring a priori voting power is not descriptive but prescriptive; not empirical but normative. It is indispensable in the proper constitutional design and assessment of decision rules. Here, it is important to quantify the voting power each member is granted by the rule itself."* Some further findings of computationally favorable assumptions for the Banzhaf power index computations can be found in Algaba et al. (2003).

2. Base concepts

Suppose we have a set $N = \{1, \ldots, n\}$, $n \in \mathbb{N}$. This set will represent the set of voters (voting bodies, parties, countries, etc.), so that each voter is represented by just one index from N. We will not consider the elements of N to be further divisible. Suppose a vector space $\mathbb{V}^n$ above the field of real numbers and the set $\mathbb{S}^n \subset \mathbb{V}^n$ of all vectors $\mathbf{w} = (w_1, \ldots, w_n)$, such that $\sum_{k=1}^{n} w_k = 1$ and $w_k \geq 0$ for any $k \in \{1, \ldots, n\}$. The set of all real numbers between 0 and 1 (including both) we denote Λ. The ordered couples $(\lambda, \mathbf{w}) \in \Lambda \times \mathbb{S}^n$ we call committees. The vector $\mathbf{w}$ from $\mathbb{S}^n$ we call a vector of weights, and the number λ from Λ we call a quota. Suppose an n-dimensional unit cube and denote the set of all its vertices $\mathbb{C}^n$, i.e. $\mathbb{C}^n = \{(c_1, \cdots, c_n) : c_i \in \{0,1\}, i \in \{1, ..., n\}\}$. The cardinality of $\mathbb{C}^n$ is clearly 2^n. A proposition is approved if, and only if, the sum of the weights of those voters who accept the proposition is greater than or equal to the quota, i.e.:

$$\sum_{k=1}^{n} w_i c_i \geq \lambda, \tag{1}$$

where $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{C}^n$ is defined as $c_i = 0$ if the i-th voter rejected the proposition and $c_i = 1$ if he or she approved it. We call a coalition any subset Q of N such that $j \in Q \Leftrightarrow c_j = 1$. Each $\mathbf{c} \in \mathbb{C}^n$ represents just one coalition. We say, the coalition $Q \subset N$ is winning, when (1) and $j \in Q \Leftrightarrow c_j = 1$ are fulfilled. The efficiency of a committee $(\lambda, \mathbf{w}) \in \Lambda \times \mathbb{S}^n$ is defined as the number of winning coalitions divided by the number of all possible coalitions:

$$\varepsilon(\lambda, \mathbf{w}) = \frac{\sum_{\mathbf{c} \in \mathbb{C}^n} I\left[\sum_{i=1}^{n} w_i c_i \geq \lambda\right]}{2^n}, \tag{2}$$

where the $I[A]$ is the identifier of A, i.e. $I[A] = 1$ if, and only if, condition A is true, $I[A] = 0$ otherwise.

We are interested in finding the maximum efficiency (resp. minimum efficiency) with respect to the weights given the quota and number of voters. So we are searching for m_λ^n and τ_λ^n given by (3) and (4):

$$m_\lambda^n := \max_{\mathbf{w}\in\mathbb{S}^n} \varepsilon(\lambda, \mathbf{w}), \tag{3}$$

$$\tau_\lambda^n := \min_{\mathbf{w}\in\mathbb{S}^n} \varepsilon(\lambda, \mathbf{w}). \tag{4}$$

3. Basic findings

Lemmas 1 and 2 are simple observations that we provide without rigorous proof and which provide us with the efficiency maxima and minima for certain quotas without any computations.

Lemma 1. *The maximum efficiency for a quota higher than* $\frac{1}{2}$ *is* $\frac{1}{2}$ *for any size committee* $n \geq 2$.

Lemma 2. *The minimum efficiency for a quota lower than* $\frac{1}{2}$ *is* $\frac{1}{2}$ *for any size committee* $n \geq 2$.

Lemma 3 is used in the proof of Lemma 5. For many quotas it enables the lowering of the number of computations needed to be performed in order to find the efficiency maximum, resp. efficiency minimum.

Lemma 3. *Let* $n \geq 2$, $\mathbf{w} = (w_1, \ldots, w_n)$ *and* $0 < \lambda \leq 1$. *Then at least*

$$\max\left\{0, n - \left\lfloor \frac{1}{\lambda} \right\rfloor\right\}$$

weights from the set $\{w_1, \ldots, w_n\}$ *are lower than* λ *for any set of weights.*

Proof. There are two possibilities:

(i) $n \leq \lfloor 1/\lambda \rfloor$. We have $1/n \geq \lambda/(\lfloor 1/\lambda \rfloor \lambda) = \lambda/(1-p)$, where $\lambda > p \geq 0$ and hence $1/n \geq \lambda$. So, there exists such a set of weights where none is lower than λ. These weights are, for example, the weights $(1/n, \ldots, 1/n)$. This is consistent with the statement as $\max\{0, n - \lfloor 1/\lambda \rfloor\} = 0$.

(ii) $n > \lfloor 1/\lambda \rfloor$. We will verify the proof by contradiction. Assume that the statement of the lemma does not hold. There would be exactly $n - \lfloor 1/\lambda \rfloor - m$ weights lower than λ, for some m such that $0 < m \leq n - \lfloor 1/\lambda \rfloor$, $m \in \mathbb{N}$. Then there must be exactly $\lfloor 1/\lambda \rfloor + m$ weights higher than or equal to λ, and for the sum W of all these weights that are higher than or equal to λ must hold:

$$W \geq \lambda\left(\left\lfloor \frac{1}{\lambda} \right\rfloor + m\right) \geq \lambda\left(\left\lfloor \frac{1}{\lambda} \right\rfloor + 1\right) \geq 1 - p + \lambda > 1,$$

because $\lambda \lfloor 1/\lambda \rfloor = 1 - p$, where $0 \leq p < \lambda$. But this is simply in contradiction with $\sum_{i=1}^n w_i = 1$, $w_i \geq 0$. Hence the statement of the lemma must hold. □

4. Maximizing and minimizing efficiency with respect to weights

The following theorem can be used for maximum and minimum efficiency computations, given the quotas and numbers of voters.

Theorem 1. *Suppose $n \in \mathbb{N}$, $\lambda \in \Lambda$ and $\mathbf{w} \in \mathbb{S}^n$. Then the maximum as well as minimum of the voting efficiency $\varepsilon(\lambda, \mathbf{w})$ is attained in at least one of the points from the set:*

$$\left\{\left(\lambda, \left(\underbrace{\frac{1}{n}, \dots, \frac{1}{n}}_{n-\text{times}}\right)\right), \left(\lambda, \left(0, \underbrace{\frac{1}{n-1}, \dots, \frac{1}{n-1}}_{(n-1)-\text{times}}\right)\right), \dots, \right.$$
$$\left.\left(\lambda, \left(\underbrace{0, \dots, 0}_{(n-2)-\text{times}}, \frac{1}{2}, \frac{1}{2}\right)\right), \left(\lambda, \left(\underbrace{0, \dots, 0}_{(n-1)-\text{times}}, 1\right)\right)\right\} \tag{5}$$

The cardinality of this set is n.

Proof. At first we express the problem as a maximization problem of integer programming:

$$\max_{\mathbf{d} \in \mathbb{R}^{2^n}, \mathbf{w} \in \mathbb{R}^n} \mathbf{1}^{\mathrm{T}}\mathbf{d} + \mathbf{0}^{\mathrm{T}}\mathbf{w} \tag{6}$$

subject to :

$$\bar{\mathbf{1}}^{\mathrm{T}}\mathbf{w} = 1 \tag{7}$$

$$d_i\left(\mathbf{w}^{\mathrm{T}}\mathbf{c}^i - \lambda\right) \geq 0 \tag{8i}$$

$$\text{for all } i \in \{1, \dots, 2^n\}$$

$$\mathbf{w} \geq \mathbf{0} \tag{9}$$

$$w_1 \leq w_2 \leq \ldots \leq w_n \tag{10}$$

$$\mathbf{d} \in \{0, 1\}^{2^n} \tag{11}$$

where $\mathbf{1}$ is an $2^n \times 1$-type vector of ones, $\bar{\mathbf{1}}$ is an $n \times 1$-type vector of ones, $\mathbf{w}$ is an $n \times 1$-type vector of real numbers, $\mathbf{d}$ is an $2^n \times 1$-type vector of real numbers and $0 \leq \lambda < 1$ is a real number, and $\mathbf{c}^i$ stands for the vector representing a vertex of an n-dimensional unit cube such that $\mathbf{c}^i \neq \mathbf{c}^j \Leftrightarrow i \neq j$, for any $i, j \in \{1, \dots, 2^n\}$. By changing condition (11) to $\mathbf{d} \in [0, 1]^{2^n}$ (which we denote (11')), we enlarge the set of all feasible solutions, but the optimal solution $(\mathbf{d}^*, \mathbf{w}^*)$ remains the same for whatever $i \in \{1, \dots, 2^n\}$. If $0 < d_i^* < 1$, then the value of the objective function would increase when shifting the value of d_i^* to 1 while all the conditions hold (because $0 < d_i^*$, we also have $\mathbf{w}^{*\mathrm{T}}\mathbf{c}^i \geq \lambda$ and so (8i) holds). Therefore, whenever $0 < d_i^* < 1$, the solution $(\mathbf{d}^*, \mathbf{w}^*)$ cannot be optimal. Let's denote $\mathcal{F}$ the set of all feasible solutions—i.e. all the vectors $(\mathbf{d}, \mathbf{w})$, for which the conditions (7), (8i) for all $i \in \{1, \dots, 2^n\}$, (9), (10) and (11') hold. Now we prove the set $\mathcal{F}$ is convex. Assume, we have two different vectors from $\mathcal{F}$, let's denote them $\left(\mathbf{d}^{(1)}, \mathbf{w}^{(1)}\right)$ and $\left(\mathbf{d}^{(2)}, \mathbf{w}^{(2)}\right)$. We need to show that

$$\alpha\left(\mathbf{d}^{(1)}, \mathbf{w}^{(1)}\right) + (1-\alpha)\left(\mathbf{d}^{(1)}, \mathbf{w}^{(1)}\right) \in \mathcal{F}.$$

We easily confirm this fact by checking that the set given by each single constraint[3] is convex and use the fact, that any intersection of a finite number of convex sets is also a convex set.

(i) For vectors fulfilling (7):

The set of all vectors $(\mathbf{d}, \mathbf{w}) \in \mathbb{R}^{2^n+n}$ which are given by (7) is convex since having any two different vectors $\left(\mathbf{d}^{(1)}, \mathbf{w}^{(1)}\right)$ and $\left(\mathbf{d}^{(2)}, \mathbf{w}^{(2)}\right)$, for which

$$\sum_{j=1}^{n} w_j^{(1)} = 1 \text{ and } \sum_{j=1}^{n} w_j^{(2)} = 1$$

we have for any real $0 \leq \alpha \leq 1$

$$\alpha \mathbf{w}^{(1)} + (1-\alpha)\mathbf{w}^{(2)} = \alpha\left(\mathbf{w}^{(1)} - \mathbf{w}^{(2)}\right) + \mathbf{w}^{(2)}$$

and so

$$\bar{\mathbf{1}}^{\mathrm{T}}\left[\alpha\left(\mathbf{w}^{(1)} - \mathbf{w}^{(2)}\right) + \mathbf{w}^{2}\right] = \alpha - \alpha + 1 = 1.$$

So each convex combination of any two vectors from $\mathcal{F}$ fulfilling (7) also fulfills (7) as well.

(ii) For vectors fulfilling (8i):

The set of all vectors $(\mathbf{d}, \mathbf{w}) \in \mathbb{R}^{2^n+n}$ which are given by (8i) is convex, because having any two different vectors $\left(\mathbf{d}^{(1)}, \mathbf{w}^{(1)}\right)$ and $\left(\mathbf{d}^{(2)}, \mathbf{w}^{(2)}\right)$, for which

$$d_i^{(1)}\left(\sum_{j=1}^{n} w_j^{(1)} c_j^i - \lambda\right) \geq 0 \text{ and } d_i^{(2)}\left(\sum_{j=1}^{n} w_j^{(2)} c_j^i - \lambda\right) \geq 0$$

we have for any real $0 \leq \alpha \leq 1$

$$\left(\alpha d_i^{(1)} + (1-\alpha) d_i^{(2)}\right)\left[\sum_{j=1}^{n}\left(\alpha w_j^{(1)} + (1-\alpha) w_j^{(2)}\right) c_j^i - \lambda\right] =$$

$$\alpha d_i^{(1)}\left[\sum_{j=1}^{n}\left(\alpha w_j^{(1)}\right) c_j^i - \alpha\lambda\right] + (1-\alpha) d_i^{(2)}\left[\sum_{j=1}^{n}\left((1-\alpha) w_j^{(2)}\right) c_j^i - (1-\alpha)\lambda\right] +$$

[3] Each single constraint from the set of constraints $\{(7), \{(8\mathrm{i}), i \in \{1, \ldots, 2^n\}\}, (9), (10), (11')\}$.

$$\alpha d_i^{(1)} \sum_{j=1}^{n} (1-\alpha) w_j^{(2)} c_j^i + (1-\alpha) d_i^{(2)} \sum_{j=1}^{n} \alpha w_j^{(1)} c_j^i =$$

$$\underbrace{\alpha^2 d_i^{(1)} \left[\sum_{j=1}^{n} w_j^{(1)} c_j^i - \lambda\right]}_{\geq 0} + \underbrace{(1-\alpha)^2 d_i^{(2)} \left[\sum_{j=1}^{n} w_j^{(2)} c_j^i - \lambda\right]}_{\geq 0} +$$

$$\underbrace{\alpha d_i^{(1)} \sum_{j=1}^{n} (1-\alpha) w_j^{(2)} c_j^i}_{\geq 0} + \underbrace{(1-\alpha) d_i^{(2)} \sum_{j=1}^{n} \alpha w_j^{(1)} c_j^i}_{\geq 0} \geq 0.$$

Hence each convex combination of any two vectors from $\mathcal{F}$ fulfilling (8i) fulfills (8i) as well.

(iii) For vectors fulfilling (9):

The set of all vectors $(\mathbf{d}, \mathbf{w}) \in \mathbb{R}^{2^n+n}$ which are given by (9) is convex, because any convex combination of two non-negative vectors is a non-negative vector. Hence each convex combination of any two vectors from $\mathcal{F}$ fulfilling (9) fulfills (9) as well.

(iv) For vectors fulfilling (10):

The set of all vectors $(\mathbf{d}, \mathbf{w}) \in \mathbb{R}^{2^n+n}$ which are given by (10) is convex, because having any two different vectors $\left(\mathbf{d}^{(1)}, \mathbf{w}^{(1)}\right)$ and $\left(\mathbf{d}^{(2)}, \mathbf{w}^{(2)}\right)$, for which $w_1^{(1)} \leq w_2^{(1)} \leq \ldots \leq w_n^{(1)}$ and $w_1^{(2)} \leq w_2^{(2)} \leq \ldots \leq w_n^{(2)}$, we get $\alpha w_1^{(1)} + (1-\alpha) w_1^{(2)} \leq \alpha w_2^{(1)} + (1-\alpha) w_2^{(2)} \leq \ldots \leq \alpha w_n^{(1)} + (1-\alpha) w_n^{(2)}$. Hence each convex combination of any two vectors from $\mathcal{F}$ fulfilling (10) fulfills (10) as well.

(v) For vectors fulfilling (11'):

The set of all vectors $(\mathbf{d}, \mathbf{w}) \in \mathbb{R}^{2^n+n}$ which are given by (11') is convex as the convex combination of two real numbers in $[0,1]$ is in $[0,1]$. Hence each convex combination of any two vectors from $\mathcal{F}$ fulfilling (11') fulfills (11') as well.

So we are maximizing a linear function on a convex set and hence, the solution to the problem must be on the frontier[4] of the set of all feasible solutions $\mathcal{F}$. We easily find out that[5]

$$w_i \leq \frac{1}{n-i+1}.$$

Let's have a vector $(\mathbf{d}^*, \mathbf{w}^*)$ that maximizes the objective function. We show this vector can be expressed as a convex combination of vectors from (5). The convex combination

[4] We show $\mathcal{F}$ is a convex polytope and hence the optimal solution will be in an extreme point.

[5] It is due to condition (10) or otherwise the weights would add up to a number strictly higher than 1.

parameters $0 \leq a_k \leq 1$, $k = 1, \ldots, n$ can be explicitly computed by solving the following system of equations:[6]

$$\sum_{k=n+1-r}^{n} \frac{a_k}{k} = w_r^*, \text{ for } r = 1, \ldots, n. \tag{12}$$

Each a_k represents the parameter in a convex combination that corresponds to the vector

$$\left(\mathbf{d}^*, 0, \ldots, 0, \underbrace{\frac{1}{k}, \ldots, \frac{1}{k}}_{k-\text{times}} \right).$$

We can compute the a_k, $k = 1, \ldots, n$ from the w_k^*, $k = 1, \ldots, n$ using the Gauss elimination.

$$\begin{array}{rcl} a_1 & = & \left(w_n^* - w_{n-1}^*\right) \cdot 1 \\ a_2 & = & \left(w_{n-1}^* - w_{n-2}^*\right) \cdot 2 \\ & & \vdots \\ a_{n-1} & = & \left(w_2^* - w_1^*\right) \cdot (n-1) \\ a_n & = & w_1^* \cdot n \end{array} \tag{13}$$

Now we have

$$\mathbf{w}^* = \sum_{k=1}^{n} a_k \cdot \left(0, \ldots, 0, \underbrace{\frac{1}{k}, \ldots, \frac{1}{k}}_{k-\text{times}} \right),$$

which is a convex combination of the vectors from (5) as $\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} w_k^* = 1$ and $a_k \geq 0$ for all $k = 1, \ldots, n$. Finally the vector $(\mathbf{d}^*, \mathbf{w}^*)$ can be expressed as a convex combination of the vectors from (5) because, of course, $d_i^* \sum_{k=1}^{n} a_k = d_i^*$.

When minimizing the objective function on set $\mathcal{F}$, the optimal solution must be also attained in set (5). $\square$

This theorem tells us that to find the maximum efficiency of a given quota is a very easy and fast exercise. We need only to check the n values to find out the maximum efficiency for any quota. In fact, we don't even have to do that, as we will see later on. In the following lemma we use Theorem 1 and express the maximum and minimum efficiency explicitly as a function of the quota and the number of voters.

Lemma 4. *Assume the quota* $0 < \lambda \leq 1$ *and the number of voters* $n \in \mathbb{N}$, $n > 1$. *Then*

$$m_\lambda^n = \frac{1}{2^n} \max_{i \in \{0, \ldots, n-1\}} \left\{ 2^i \cdot \sum_{j=\lceil (n-i)\lambda \rceil}^{n-i} \binom{n-i}{j} \right\} \tag{14}$$

[6] For example by the Gauss elimination method it is very easy since the system matrix is already triangular.

and

$$\tau_\lambda^n = \frac{1}{2^n} \min_{i \in \{0,\dots,n-1\}} \left\{ 2^n - 2^i \cdot \sum_{j=\lfloor (n-i)(1-\lambda) \rfloor +1}^{n-i} \binom{n-i}{j} \right\}. \tag{15}$$

Proof. First we prove the maximization part. The proof is easy once we know the efficiency maximum is attained in set (5). The number of winning coalitions of size $0 < k \leq n$ when considering the vector is

$$\left(\underbrace{0,\dots,0}_{i\text{-times}}, \underbrace{\frac{1}{n-i},\dots,\frac{1}{n-i}}_{(n-i)\text{-times}} \right) \tag{16}$$

with no zero weight is $\binom{n-i}{k}$, but only if $k \geq \lceil (n-i)\lambda \rceil$. So in total, there are

$$\sum_{k=\lceil (n-i)\lambda \rceil}^{n-i} \binom{n-i}{k}$$

winning coalitions made of voters with non-zero weights. For each winning coalition made of voters with non-zero weights, there are $2^i - 1$ other winning coalitions, including the voters with zero weights, since we have to add all the subsets of voters with zero weights, except the empty set.[7] There are, in total,

$$2^i \cdot \sum_{k=\lceil (n-i)\lambda \rceil}^{n-i} \binom{n-i}{k}$$

winning coalitions corresponding to vector (16). When we compute this value for all the $i = 0,\dots,n-1$ and we take the highest of them, we must have the maximum attainable number of winning coalitions. To compute the maximum efficiency, we just divide it by 2^n.

Now we prove the minimization part. We know from Theorem 1 that the minimum will be attained for the vector of weights from set (5) and hence we will further only consider the vectors from (5). Let's denote the sum of the weights of one of the winning coalitions $W \geq \lambda$. Then clearly $1 - W \leq 1 - \lambda$. $1 - W$ is the sum of the weights of all the voters in a complementary coalition.[8] For each winning coalition, for committees with quota λ, there is only one complementary coalition with a sum of weights lower than or equal to $1 - \lambda$. Assume we have such a vector of weights that minimizes the number of winning coalitions for the quota λ. Then we also have the minimum number of complementary coalitions with a sum of weights lower than or equal to $1 - \lambda$. Since

[7] Because the empty set represents the coalition made only of the voters with non-zero weights and this coalition has already been counted.

[8] We call a complementary coalition such a coalition that consists of all the voters who are not in the original coalition.

in total there are always 2^n coalitions, minimizing the number of coalitions with a sum of weights lower than or equal to $1-\lambda$, with respect to the weights, is the same as maximizing the number of coalitions with a sum of weights over $1-\lambda$ with respect to the weights. In total, there are

$$\sum_{k=\lfloor (n-i)(1-\lambda)\rfloor+1}^{n-i} \binom{n-i}{k}$$

winning coalitions made of voters with non-zero weights for a vector of weights

$$\left(\underbrace{0,\dots,0}_{i\text{-times}},\underbrace{\frac{1}{n-i},\dots,\frac{1}{n-i}}_{(n-i)\text{-times}}\right). \tag{17}$$

Analogously to the first part of the proof, the total number of winning coalitions corresponding to vector (17) from set (5) is

$$2^n - 2^i \cdot \sum_{k=\lfloor (n-i)(1-\lambda)\rfloor+1}^{n-i} \binom{n-i}{k}.$$

Computing this for each vector from (5), we obtain this value for $i=0,\dots,n-1$ and by taking the minimum, we obtain the minimum number of winning coalitions under the quota λ. To compute the minimum efficiency, we just divide it by 2^n. □

The previous lemma shows how to compute the minimum and maximum attainable efficiency given quota λ and number of voters n. However, as we mentioned, we need not compute all the n values to find the maximum and n values to find the minimum efficiency. In the following lemma we show it is not necessary to compute all the n values and that it suffices to compute just the $n-\min\{n,\lfloor 1/\lambda\rfloor\}+1$ values. This means we can compute less than n values to find the efficiency maximum whenever $\lambda \le 1/2$, resp. the efficiency minimum, whenever $\lambda > 1/2$. But according to Lemmas 1 and 2, these are the only situations that are of any interest.

Lemma 5. *Assume the quota is $0<\lambda\le 1$ and the number of voters is $n\in\mathbb{N}$, $n>1$. Then*

$$m_\lambda^n = \frac{1}{2^n} \max_{i\in\left\{0,\dots,n-\min\left\{n,\left\lfloor \frac{1}{\lambda}\right\rfloor\right\}\right\}} \left\{ 2^i \cdot \sum_{k=\lceil (n-i)\lambda\rceil}^{n-i} \binom{n-i}{k} \right\} \tag{18}$$

and

$$\tau_\lambda^n = \frac{1}{2^n} \min_{i\in\left\{0,\dots,n-\min\left\{n,\left\lfloor \frac{1}{1-\lambda}\right\rfloor\right\}\right\}} \left\{ 2^n - 2^i \cdot \sum_{k=\lfloor (n-i)(1-\lambda)\rfloor+1}^{n-i} \binom{n-i}{k} \right\}. \tag{19}$$

Proof. First we prove the part about the maximum efficiency. Within this proof, we will always take into account only the vectors of weights from (5). Two situations can occur. In the first, no weight is higher than or equal to λ. In the second, at least one weight is higher than or equal to λ. Now, we are interested only in the second situation. Because we only have vectors from (5), at least one weight higher than λ is the same as any non-zero weight that is higher than or equal to λ. We need to specify for which vectors from (5) are all the non-zero weights higher than or equal to λ. From lemma 3 we know that at least $\max\{0, n - \lfloor 1/\lambda \rfloor\}$ must be lower than λ. This implies that no more than $n - \max\{0, n - \lfloor 1/\lambda \rfloor\} = \min\{n, \lfloor 1/\lambda \rfloor\}$ weights can be higher than or equal to λ. Assume there is exactly $\hat{k} = \min\{n, \lfloor 1/\lambda \rfloor\}$ weights higher than or equal to λ—this means we assume the vector

$$\left(0, \ldots, 0, \underbrace{\frac{1}{\min\{n, \lfloor \frac{1}{\lambda} \rfloor\}}, \ldots, \frac{1}{\min\{n, \lfloor \frac{1}{\lambda} \rfloor\}}}_{\min\{n, \lfloor \frac{1}{\lambda} \rfloor\}\text{-times}}\right)$$

from (5). Indeed,

$$\min\left\{n, \left\lfloor \frac{1}{\lambda} \right\rfloor\right\} \geq \lambda.$$

Hence for any $\hat{k} < \min\{n, \lfloor 1/\lambda \rfloor\}$ is $1/\hat{k} > \lambda$, and so any non-zero weight is also higher than λ, and the efficiency equals

$$\frac{\sum_{j=n-\hat{k}}^{n-1} 2^j}{2^n},$$

as there are 2^{n-1} coalitions containing the first non-zero weight, 2^{n-2} coalitions containing the second non-zero weight which do not contain the first non-zero weight, 2^{n-3} coalitions containing the third non-zero weight which contain neither the first, nor the second non-zero weight, and so on up to the $\hat{k}$-th non-zero weight. In total there are exactly

$$\sum_{j=n-\hat{k}}^{n-1} 2^j = 2^n - 2^{n-\hat{k}} \tag{20}$$

winning coalitions.

As the sum (20) is strictly increasing in $\hat{k}$, we may compute the number of winning coalitions only for $\hat{k} = \min\{n, \lfloor 1/\lambda \rfloor\}, \ldots, n$ as for any $\hat{k} < \min\{n, \lfloor 1/\lambda \rfloor\}$, the efficiency is surely strictly lower than the one for $\hat{k} = \min\{n, \lfloor 1/\lambda \rfloor\}$.

$\hat{k}$ stands for the number of non-zero weights, while i in the expression (14) corresponds to the number of zero weights. The number of zero weights can be expressed as $n - \hat{k}$. So we use equation (14), but we take the maximum just of the efficiency for the vectors of weights from (5), which contain up to $\max\{0, n - \lfloor 1/\lambda \rfloor\}$ zero weights. By doing this, we obtain (18).

Now we prove the part about the minimum efficiency using the equality (21)

$$\frac{1}{2^n} \min_{i \in \{0, \ldots, n - \min\{n, \lfloor \frac{1}{1-\lambda} \rfloor\}\}} \left\{ 2^n - 2^i \cdot \sum_{k=\lfloor (n-i)(1-\lambda) \rfloor + 1}^{n-i} \binom{n-i}{k} \right\} = \tag{21}$$

$$1-\frac{1}{2^n}\max_{i\in\left\{0,\ldots,n-\min\left\{n,\left\lfloor\frac{1}{1-\lambda}\right\rfloor\right\}\right\}}\left\{2^i\cdot\sum_{k=\lfloor(n-i)(1-\lambda)\rfloor+1}^{n-i}\binom{n-i}{k}\right\}.$$

We denote $\mu := 1-\lambda$ and use a slightly modified Lemma 4 for a quota equal to μ. The modification is in requiring the sum of weights being strictly above the quota. In other words, we maximize the number of coalitions with a sum of weights strictly above $1-\lambda$ and then subtract this maximized number from the total number of coalitions, obtaining the minimum number of coalitions with a sum of weights higher than or equal to λ. □

5. The symmetry of efficiency

Lemma 6. *Assume $n \geq 2$ and $0 < \lambda \leq 1$. Then*

$$\tau_\lambda^n = 1 - m_{1-\lambda+\varepsilon}^n, \tag{22}$$

for any ε such that $\lambda > \varepsilon \geq 0$, and ε is smaller than the smallest difference between any two distinct sums of weights of two distinct coalitions.

Proof. Consider all the winning coalitions for quota β and denote the number of them k. Clearly, for each coalition there exists a unique complementary coalition made of all the voters who are not in the original winning coalition. Hence the number of winning coalitions for quota β is the same as the number of all coalitions that are made of voters whose weights add up to at most $1-\beta$. When k is the maximal attainable number of winning coalitions for quota β, then it is also the maximal attainable number of coalitions with voters whose weights add up to at most $1-\beta$. Because there is always just 2^n of all coalitions, maximizing the number of all coalitions with a total weight lower than or equal to $1-\beta$ is the same as minimizing the number of all coalitions with a total weight strictly higher than β. Assume now we have added a very small number ε to quota β such that the maximal number of winning coalitions have been lowered only by the coalitions with a total weight equal to β (if there were any). Then the maximum number of winning coalitions K for quota $\beta+\varepsilon$ is equal to the maximum number of all losing coalitions for quota $1-\beta$, which is the same as the minimum of $2^n - K$ of coalitions winning for quota $1-\beta$. So we have $\tau_{1-\beta}^n = 1 - m_{\beta+\varepsilon}^n$. Denoting $\lambda = 1-\beta$ as $0 \leq \beta \leq 1$ we obtain $\tau_\lambda^n = 1 - m_{1-\lambda+\varepsilon}^n$. □

It is clear the problem is not fully symmetric around the quota equal to $1/2$ since the coalition is winning whenever the sum of its voters' weights is at least equal to the quota, which means higher than or equal to the quota. The slight asymmetry is caused by reaching the quota being sufficient for the coalition to win.

6. Efficiency structure

In this section, we analyze the structure of the efficiency maxima and minima with respect to weights when we take the maximum (resp. minimum) as a function of the quota. We show there are at most asymptotically $3n^2/2\pi^2 + o(n\log(n))$ different efficiency maxima (resp. efficiency minima) that can be attained when moving the quota within Λ (n stands for the number of voters). We also show in which intervals the quota can move freely without changing the efficiency maximum (resp. efficiency minimum).

For k voters, $k \in \mathbb{N}$, we divide the interval $(0,1)$ in the following way: We denote $d_{ij} = i/(k-j)$ for $i \in \mathbb{N}, 0 \leq i \leq (k-j)$, $j = 0,...,k-1$ and sort all the d_{ij}'s in nondecreasing order. We obtain some finite sequence (d_i) of numbers within the interval $[0,1]$ sorted in nondecreasing order. Now we create a subsequence $\left(\hat{d}_i\right)$ of the sequence (d_i) by dropping the elements which are not unique, except for just one. In other words, whenever there are at least two equivalent elements of the sequence (d_i) we drop all of them except just one. In the final sequence $\left(\hat{d}_i\right)$ each value is hence unique and the sequence becomes sorted in ascending order and contains all the values that were in the original sequence (d_i). The elements of the final sequence $\left(\hat{d}_i\right)$ we denote $0 = \hat{d}_1 < \hat{d}_2 < \cdots < \hat{d}_{\hat{k}} = 1$ and call them milestones. The interval $(0,1)$ is then divided as follows

$$(0,1) = \left(\hat{d}_1, \hat{d}_2\right) \cup \left[\hat{d}_2, \hat{d}_3\right) \cup \cdots \cup \left[\hat{d}_{\hat{k}-1}, \hat{d}_{\hat{k}}\right). \tag{23}$$

Lemma 7. *The number of milestones (when we include* 0 *and* 1*) for n voters in the interval* $[0,1]$ *is* $k_n = 2 + \sum_{j=1}^{n} d_j$, *where* $d_j = j - 1 - \sum_{\{p: p/j \wedge p<j \wedge p>0\}} d_p$.

Proof. The proof can be accomplished by mathematical induction. We say the k-th step will be the process of computing the number of milestones for a denominator equal to k. In each step those milestones are added which are not equal to milestones from any of the previous steps. The milestones from the k-th step are equal to milestones added in the l-th step, when $l < k$ if and only if there is a number $p \in \mathbb{N}$ so that p divides k and l. To the total number of milestones we add 2, representing 0 and 1. Because the sequence of milestones in the literature is known as the Farey sequence and the cardinality of the set of all the elements of the sequence is expressed using the Euler's totients function, we will not present a rigorous detailed proof here. $\square$

An interesting observation is that the number of milestones for n voters is asymptotically equal to $3n^2/\pi^2 + o(n\log(n))$, see for example p. 155 in Vardi (1991). As the efficiency maximum for quotas above one half is just a single value (see Lemma 1) and the efficiency minimum for quotas below one half is also just a single value (see Lemma 2), we see that at most just about half of the milestones are in fact relevant[9] for the efficiency maximum, or minimum, to change. Hence there is asymptotically at most $3n^2/2\pi^2 + o(n\log(n))$ of a different efficiency maxima (resp. efficiency minima) that can be attained when moving the quota within Λ.

Any two milestones such that there is no other milestone between them we will call adjacent milestones (Fareys pairs). Having two adjacent milestones $\hat{d}_i$ and $\hat{d}_{i+1}$ we will call $\hat{d}_i$ the lower adjacent milestone with respect to $\hat{d}_{i+1}$, and $\hat{d}_{i+1}$ the higher adjacent milestone with respect to $\hat{d}_i$.

Lemma 8. *The efficiency maximum (resp. efficiency minimum) is the same for all the quotas between two adjacent milestones, or for quotas such that the higher quota is*

[9] As for each milestone, we can find another one which has the same distance from $1/2$.

equal to some milestone $\hat{d}_j$ and the lower quota is strictly above the adjacent lower milestone with respect to $\hat{d}_j$.

Proof. Assume we have two quotas $0 < \lambda_1 < \lambda_2 < 1$ and k different milestones. Assume there is an index $i \in \{1,\dots,k-1\}$ such that $\hat{d}_i < \lambda_1 < \lambda_2 \le \hat{d}_{i+1}$. In other words, both the quotas are between two adjacent milestones, or the higher quota is equal to a milestone, while the lower quota is strictly above the adjacent lower milestone. This implies there is no milestone strictly between λ_1 and λ_2. From the definition of a milestone we know there is also no rational number with a denominator equal to or lower than n strictly between the two quotas. From Lemma 5 we know the efficiency maximum (resp. efficiency minimum) does not depend directly on the quota λ, but on the expression $\lceil \lambda j \rceil$ for some $j \in \{1,\dots,n\}$. If for every $j \in \{1,\dots,n\}$ is $\lceil \lambda_1 j \rceil = \lceil \lambda_2 j \rceil$, then the efficiency maximum (resp. efficiency minimum) surely is the same for both quotas.

We make the proof by contradiction in assuming the efficiency maximum (resp. efficiency minimum) is different for quota λ_1 and for quota λ_2. Then there must be some $j \in \{1,\dots,n\}$ so that $\lceil j\lambda_1 \rceil < \lceil j\lambda_2 \rceil$, as if such a j would not exist then the efficiency maximum (resp. efficiency minimum) could not be different for the two quotas. There must exist a positive integer p, such that $\lceil j\lambda_2 \rceil = \lceil j\lambda_1 \rceil + p$. However, for us it is sufficient to know that

$$\lceil j\lambda_2 \rceil \ge \lceil j\lambda_1 \rceil + 1. \tag{24}$$

We will prove the statement

$$\lambda_1 < \frac{\lfloor j\lambda_2 \rfloor}{j} \le \lambda_2.$$

The inequality $\lfloor j\lambda_2 \rfloor \le j\lambda_2$ is easy, as $\lfloor\cdot\rfloor$ stands for the highest lower integer. To prove the strict inequality $j\lambda_1 < \lfloor j\lambda_2 \rfloor$ is a little bit more challenging. From (24) we know that $j\lambda_1 \le \lceil j\lambda_1 \rceil \le \lceil j\lambda_2 \rceil - 1 \le \lfloor j\lambda_2 \rfloor$. However, if $j\lambda_1 = \lfloor j\lambda_2 \rfloor$, then $j\lambda_1 \in \mathbb{N}$ and hence $j\lambda_1/j = \lambda_1$ must be a milestone. But this would be in contradiction with $\hat{d}_i < \lambda_1 < \hat{d}_{i+1}$. So, in total we get $\lambda_1 < \lfloor j\lambda_2 \rfloor / j$.

We have proven, that

$$\lambda_1 < \frac{\lfloor j\lambda_2 \rfloor}{j} \le \lambda_2,$$

but if $\lfloor j\lambda_2 \rfloor / j = \lambda_2$, we are not yet in contradiction with the statement of the lemma as λ_2 is allowed to be a milestone. If $\lfloor j\lambda_2 \rfloor / j < \lambda_2$, we are finished, as $\lfloor j\lambda_2 \rfloor / j$ is surely a milestone and it is strictly between the two quotas. To show it is a milestone, it suffices to recall that a milestone is each rational number between 0 and 1 with a denominator equal to an integer lower than or equal to n.

Now we finish the proof by solving the case $\lfloor j\lambda_2 \rfloor / j = \lambda_2$ (which means that λ_2 is a milestone). If this holds, we have $j\lambda_2 \in \mathbb{N}$. In this case, there is a number $\lceil j\lambda_1 \rceil / j$, which is a milestone and, moreover, we can easily show that:

$$\lambda_1 \le \frac{\lceil j\lambda_1 \rceil}{j} < \lambda_2.$$

The strict inequality is simple, as $\lfloor j\lambda_2 \rfloor = j\lambda_2$ implies $j\lambda_2 = \lceil j\lambda_2 \rceil$, and we have (24) stating that $\lceil j\lambda_1 \rceil < \lceil j\lambda_2 \rceil$. The weak inequality is easy as $\lceil \cdot \rceil$ stands for the lowest higher integer.

This completes the proof, since whenever the efficiency maximum (resp. efficiency minimum) is different for quota λ_1 from the efficiency maximum for quota λ_2, we have just shown there must be a milestone between the two quotas or the higher quota must be a milestone and the lower quota must be below or equal to the adjacent lower milestone. □

This lemma tells us that whenever the quota is between any two adjacent milestones, the efficiency maximum (resp. efficiency minimum) does not change. However it does not tell us that moving the quota ”across” a milestone would always change the efficiency maximum (resp. efficiency minimum) as this is generally not true.

Theorem 1 can be used to provide information on efficiency when setting the quota for a newly designed electoral system or voting rule, but it can also be used for other economic problems, where an optimal allocation is being sought out.

7. Conclusions

We were interested in the behavior of the efficiency of a weighted voting system (probability of approval) as a function of the weights assigned to particular voters and the quota. We especially wished to find the maximal and minimal attainable efficiency with respect to the vector of weights, given the number of voters and the quota. We have answered all the raised questions:

(i) What are the minima and maxima of the efficiency function with respect to the weights and can they be expressed analytically in terms of the quota and number of voters? *This question has been answered by Theorem 1 and Lemmas 1, 2 and 5.*

(ii) Is the efficiency function symmetric in some sense around the quota equal to one half? *This question has been answered by Lemma 6.*

(iii) Are there any intervals in which the quota can move freely so that the efficiency maximum (resp. efficiency minimum) with respect to the weights does not change? If so, how they can be formally described? *This question has been answered by Lemmas 7 and 8.*

(iv) How many different values can the efficiency maximum, resp. efficiency minimum, attain in terms of the number of voters n? Can this number be expressed explicitly as a function of n? *This question has been partially answered in the text. We have noted the number of the efficiency maxima, resp. efficiency minima, asymptotically equals* $3n^2/2\pi^2 + o(n \log(n))$.

We have presented Theorem 1, allowing us to express the efficiency minimum and maximum explicitly as functions of the quota and the number of voters. Moreover, we have provided a proof of this theorem. As far as we know, there is no other work where this would have already been proven. The practical use of our theorem is mainly for control over quota-setting when designing voting rules for weighted voting systems and it can be used more generally within the context of traditional economy when searching for an optimal allocation. Usually the quota for regular propositions to be accepted by legislative institutions equals one half plus one mandate. However, constitutional quotas are often higher, since the most important democratic rules embedded in the constitutions need to be more secure than regular laws. Corollaries of our theorem help us to analyze the minimum (resp. maximum) efficiency of weighted voting systems as a function of the quota and number of voters.

Acknowledgment This research was supported by the Grant Agency of the Czech Republic, project No. 402/09/1066 "Political economy of voting behavior, rational voters' theory and models of strategic voting".

References

Algaba, E., Bilbao, J. M. Fernandez Garcia, J. R. and Lopez, J. J. (2003). Computing Power Indices in Weighted Multiple Majority Games. *Mathematical Social Sciences*, 46,63-80.

Aziz, H. and Paterson, M. (2008). Classification of Computationally Tractable Weighted Voting Games. In: World Congress on Engineering 2008, July 02–04, 2008, Imperial College London, London.

Coleman, J. S. (1971). Control of Collectivities and the Power of a Collectivity to Act. In Lieberman, B. (ed.), *Social Choice*. New York, Gordon and Breach, p. 269–300.

Felsenthal, D. S. and Machover, M. (1998). T*he Measurement of Voting Power: Theory 419 and Practice, Problems and Paradoxes*. Cheltenham, Edward Elgar.

Felsenthal, D. S. and Machover, M. (2004). A Priory Voting Power: What Is It All About? *Political Studies Review*, 2(1), 1-23.

Hosli, M. (2008). Balancing Legitimacy and Efficiency: Council Decision Rules and European Constitutional Design. *AUCO Czech Economic Review*, 2(2), 76–96.

Leech, D. and Aziz, H. (2008). The Double Majority Voting Rule of the EU Reform Treaty as a Democratic Ideal for an Enlarging Union: An Appraisal Using Voting Power Analysis. Paper presented at the European Public Choice Society meeting, Jena, March 27–30, 2008.

Leech, D., Aziz, H. and Paterson, M. (2007). Combinatorial and Computational Aspects of Multiple Weighted Voting Games. Warwick, University of Warwick, Department of Economics, The Warwick Economics Research Paper Series (TWERPS), No. 823.

Leech, D. (2002). Voting Power in the Governance of the International Monetary Fund. *Annals of Operations Research*, 109, 373–395.

Lindner, I. (2008). The Power of a Collectivity to Act in Weighted Voting Games with Many Small Voters. *Social Choice and Welfare*, 30(4), 581–601.

Matsui, T. and Matsui, Y. 2001. *NP*-completeness for Calculating Power Indices of Weighted Majority Games. *Theoretical Computer Science*, 263(1–2), 305–310.

Matsui, T. and Matsui, Y. (2000). A Survey of Algorithms for Calculating Power Indices of Weighted Majority Games. Journal of Operations Research Society of Japan, 43, 71–86.

McIntyre, E. (1954). Weighted Voting in International Organizations. *International Organization*, 8, 484–497.

Prasad, K. and Kelly, J. S. (1990). *NP*-completeness of Some Problems Concerning Voting Games. International Journal of Game Theory, 19(1), 1–9.

Vardi, I. (1991). *Computational Recreations in Mathematica*. Boston, Addison-Wesley.

Zuckerman, M., Faliszewski, P., Bachrach, Y. and Elkind, E. (2008). Manipulating the Quota in Weighted Voting Games. Proceedings of the Twenty-Third AAAI Conference on Artificial Intelligence, July 13–17, 2008, Chicago.