AUCO Czech Economic Review 5 (2011) 191–202
Acta Universitatis Carolinae Oeconomica

Skill and Chance in Insurance Policies

Vito Fragnelli[*], **Maria Erminia Marina**[**]

Received 30 July 2010; Accepted 5 February 2011

Abstract In a game theoretical setting it is possible to assign a relative measurement of the skill required to each player for maximizing his payoff in a casino game; according to this approach we analyze the relevance of skill and chance in an insurance contract, both for the insurance company and for the customer.

Keywords Skill, casino game, insurance
JEL classification C72, G22, J24

1. Introduction

In this paper we want to evaluate how much appropriate strategic choices may influence the expected payoffs of the two parts in an insurance contract. More precisely, we consider a situation in which a customer $\mathcal{C}$ wants to insure a risk R (usually represented by a non-negative random variable) with an insurance company $\mathcal{I}$; we model this situation as a non-cooperative game. The players of the game are $\mathcal{C}$ and $\mathcal{I}$; the strategies of $\mathcal{C}$ are to accept the contract only up to a fixed amount or to accept every proposal, while the strategies of $\mathcal{I}$ correspond to the possible selling prices of the policy; the payoffs of the two players depend on the evaluation of the risk and of the premium.

A similar problem, how to evaluate the relative extent to which a player can influence his payoff by using suitable strategies, was tackled by many authors, we mention Larkey et al. (1997) and more recently Borm and van der Genugten (1999, 2001) and Dreef et al. (2003, 2004). These authors analyzed zero-sum games, looking for an index that allows distinguishing between games of chance and games of skill; in particular, they tackled the real-life problem of measuring the skill necessary in order to gain in a casino game (in various countries the legislation establishes which games depend on chance and which on skill and the consequence is that the latter may be exploited freely, while the former may be exploited legally only in the casinos).

The approach of Dreef, Borm and van der Genugten consists in the analysis of three types of players: the first type (beginner) just knows how to play the game, the second type (optimal) is able, thanks to his experience, to exploit all the strategic possibilities offered by the game, the third type (fictive) is a theoretical player that has a complete knowledge both of the best strategies of the game and of the chance elements. They

[*] University of Eastern Piedmont, Department of Sciences and Advanced Technologies, Via T. Michel 11, 15121 Alessandria, Italy. Phone: +390131360224, E-mail: vito.fragnelli@mfn.unipmn.it.
[**] University of Genova, Department of Economics and Quantitative Methods, Via Vivaldi 5, 16126 Genova, Italy. Phone: +390102095247, E-mail: marina@economia.unige.it.

define the skill in a casino game comparing the increase of payoff for the optimal player w.r.t. the beginner (that allows measuring the importance of the experience), and the increase of payoff for the fictive player w.r.t. the beginner (that allows measuring the relevance of the chance elements together with the experience). They call pure games of skill those games with no influence of the random elements and pure games of chance those games in which the players experiences do not affect the outcome.

Our aim is to apply the previous approach, tailoring it to our insurance problem. Our motivation arises from the remark that at a first glance the chance plays a relevant role in subscribing an insurance policy, so we asked if also skill (experience) may be important, and if it is possible to have a measurement of its influence on the outcome of the agents. On the other hand, it is too naive to think that the insurer increases his gain asking a higher premium because he may lose the customer. Analogously, if the customer looks for a very low premium it may happen that he rejects all the proposals, keeping a high risk. Our intent is to evaluate the increasing of the gain due to a better choice of the amount of the premium that, on the one hand, the insurer has to ask for and, on the other hand, the customer accepts to pay, w.r.t. the increasing of the gain due to a theoretical knowledge of the actual amount of the risk and of the strategic behavior of the counterpart.

In our framework, we face a new problem as we have a non-zero sum game. In fact, at first Borm and van der Genugten (1999, 2001) suppose a fixed and uniform reference of the opponents against each type of player in a specific role, namely the Nash equilibrium. In a subsequent approach Dreef et al. (2003, 2004) let the opponents react optimally, depending on the type of player; this leads to the strategy of maximal opposition, as their games have zero sum. In our situation of non-zero sum games, the concept of maximal opposition has no meaning as each player looks for maximizing his gain, disregarding the payoff of the other one. In view of this, we decided to analyze the minimal gain that each type of player may guarantee himself. This approach allows us ignoring another problem that may arise with a non-zero sum game, related to multiple Nash equilibrium. In this case the different Nash equilibria correspond to different strategy profiles, in such a way that a player may choose among various strategies, ending in possible different payoffs (in a zero sum game all the Nash equilibria have the same payoff).

The organization of the paper is the following: In the next section we recall some basic definitions of non cooperative game theory; in Section 3 the insurance game is introduced; in Section 4 we recall the definition of skill introduced by Dreef, Borm and van der Genugten, with some modifications for non-zero sum games via two suitable games for fictive players; Section 5 is devoted to formulas for computing the components of the skill in the case of insurance game; Section 6 concludes.

2. Preliminary on game theory

We start by recalling some basic definitions on non cooperative games. We consider a two-person game in strategic form $G = (\Sigma_1, \Sigma_2, \pi_1, \pi_2)$ where, for $i = 1, 2$, Σ_i denotes the finite non-empty set of pure strategies of player i and $\pi_i : \Sigma_1 \times \Sigma_2 \to \mathbb{R}$ is a function that assigns to each strategy profile $(\sigma_1, \sigma_2) \in \Sigma_1 \times \Sigma_2$ the payoff of player i. A two-person game is called *zero sum* if $\pi_1 = -\pi_2$.

A mixed strategy for player $i = 1,2$ is a probability distribution over the set of his pure strategies Σ_i. We denote a mixed strategy by $p_i = (p_i(\tau))_{\tau \in \Sigma_i}$ where $p_i(\tau)$ represents the probability of choosing the pure strategy τ and by $\Delta(\Sigma_i)$ the set of mixed strategies for player $i = 1,2$. A pure strategy $\tau \in \Sigma_i$ of player $i = 1,2$ can be viewed as the mixed strategy $p_i \in \Delta(\Sigma_i)$, such that $p_i(\tau) = 1$ and $p_i(\sigma) = 0$ for each $\sigma \in \Sigma_i \setminus \{\tau\}$.

Given a mixed strategy profile $p = (p_1, p_2)$, where $p_i \in \Delta(\Sigma_i), i = 1,2$, the corresponding payoff for player $i = 1,2$ is:

$$\pi_i(p) = \sum_{(\sigma_1,\sigma_2) \in \Sigma_1 \times \Sigma_2} p_1(\sigma_1) p_2(\sigma_2) \pi_i(\sigma_1, \sigma_2)$$

The most usual solution concept for a non cooperative game is the *Nash Equilibrium (NE)*. A NE in mixed strategies is a strategy profile (p_1^*, p_2^*) such that $\pi_1(p_1^*, p_2^*) \geq \pi_1(p_1, p_2^*)$, for each $p_1 \in \Delta(\Sigma_1)$ and $\pi_2(p_1^*, p_2^*) \geq \pi_2(p_1^*, p_2)$, for each $p_2 \in \Delta(\Sigma_2)$, i.e. no player has an incentive to deviate from (p_1^*, p_2^*).

Another widely used concept is the *maxmin strategy*, where a player selects the strategy that maximizes the minimum of his possible payoffs, whatever strategy the other player chooses; so, a maxmin strategy for player 1 [2] is $\hat{p}_1 \in \Delta(\Sigma_1)$ [$\hat{p}_2 \in \Delta(\Sigma_2)$] such that:

$$\hat{p}_1 \in \operatorname*{argmax}_{p_1 \in \Delta(\Sigma_1)} \min_{p_2 \in \Delta(\Sigma_2)} \pi_1(p_1, p_2) \quad \left[\hat{p}_2 \in \operatorname*{argmax}_{p_2 \in \Delta(\Sigma_2)} \min_{p_1 \in \Delta(\Sigma_1)} \pi_2(p_1, p_2) \right]$$

or, equivalently:

$$\hat{p}_1 \in \operatorname*{argmax}_{p_1 \in \Delta(\Sigma_1)} \min_{\sigma_2 \in \Sigma_2} \pi_1(p_1, \sigma_2) \quad \left[\hat{p}_2 \in \operatorname*{argmax}_{p_2 \in \Delta(\Sigma_2)} \min_{\sigma_1 \in \Sigma_1} \pi_2(\sigma_1, p_2) \right]$$

For further details we address to Fudenberg and Tirole (1991).

Now, we consider a game associated to a stochastic situation. More precisely, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where Ω is the outcome space, $\mathcal{F}$ is a σ-algebra on Ω and $\mathbb{P}$ is a probability measure.

Let Σ_1 and Σ_2 be two finite, non-empty sets and let φ_1 and φ_2 be two functions, $\varphi_i : \Sigma_1 \times \Sigma_2 \times \Omega \to \mathbb{R}, i = 1,2$ such that for each $\sigma_1 \in \Sigma_1$ and for each $\sigma_2 \in \Sigma_2$ the function $\varphi_i(\sigma_1, \sigma_2, .) \in \mathcal{L}$, where $\mathcal{L}$ is the set of random variables defined on Ω with finite expectation. Then, we define the two-person game in strategic form $G = (\Sigma_1, \Sigma_2, \pi_1, \pi_2)$ where Σ_1 and Σ_2 are as above and $\pi_i(\sigma_1, \sigma_2) = \int_\Omega \varphi_i(\sigma_1, \sigma_2, \omega) d\mathbb{P}(\omega)$ or $\pi_i(\sigma_1, \sigma_2) = E(\varphi_i(\sigma_1, \sigma_2, \cdot)), i = 1,2$.

3. The insurance game

In this section, we consider a customer $\mathcal{C}$ that wants to insure a risk, described by a random variable $R : \Omega \to \mathbb{R}$, whose cumulative distribution function is $F : \mathbb{R} \to [0,1]$ with $F(x) = \mathbb{P}(R < x)$. Then, F is left-continuous, increasing (not necessarily strictly) and

$\lim_{x \to +\infty} F(x) = 1$; moreover we suppose that $F(x) = 0$ for each $x \leq 0$. The counterpart of the customer is an insurance company, or insurer, $\mathcal{I}$. We suppose that both players $\mathcal{C}$ and $\mathcal{I}$ are expected utility maximizers (i.e. each of them prefers a random variable X to a random variable Y if the expected utility of X is larger than the one of Y). We assume that the utility functions of the two players $u_{\mathcal{K}} : \mathbb{R} \to \mathbb{R}, \mathcal{K} = \mathcal{C}, \mathcal{I}$ are strictly increasing, concave (i.e. both of them are risk-averse) and $u_{\mathcal{K}} \circ (-R) \in \mathcal{L}, \mathcal{K} = \mathcal{C}, \mathcal{I}$.[1]

If we suppose that the company requires a premium P to buy the risk R, according to our assumptions, the contract is satisfactory for both the players if the premium satisfies the following two conditions:

$$\begin{cases} u_{\mathcal{C}}(-P) \geq E(u_{\mathcal{C}}(-R)) \\ E(u_{\mathcal{I}}(P-R)) \geq u_{\mathcal{I}}(0) \end{cases} \tag{1}$$

Let $P^{\mathcal{C}}$ and $P^{\mathcal{I}}$ be the unique solutions of the following equations, respectively:

$$u_{\mathcal{C}}(-P) = E(u_{\mathcal{C}}(-R))$$

$$E(u_{\mathcal{I}}(P-R)) = u_{\mathcal{I}}(0) \tag{2}$$

In the real-life player $\mathcal{C}$, the customer, is more risk-averse than player $\mathcal{I}$, the insurer, so under suitable hypotheses on the utility functions, it follows that $P^{\mathcal{C}} > P^{\mathcal{I}}$ and then a premium P satisfies the inequalities (1) if and only if $P^{\mathcal{C}} \geq P \geq P^{\mathcal{I}}$.

Remark 1. *The price $P^{\mathcal{I}}$ represents the theoretical minimal premium that the insurer may ask: the word theoretical is added because, as said in Goovaerts et al. (1984), the premium calculation principle defined by* (2) *provides the insurer with a measure of the risk, rather than a commercial premium (which includes some commercial loadings).*

Example 1. *We assume that the utility functions of the two players $\mathcal{C}$ and $\mathcal{I}$ are $u_{\mathcal{K}}(x) = \frac{1}{\alpha_{\mathcal{K}}}(1 - e^{-\alpha_{\mathcal{K}} x})$, $\alpha_{\mathcal{K}} > 0, \mathcal{K} = \mathcal{C}, \mathcal{I}$, with $\alpha_{\mathcal{C}} > \alpha_{\mathcal{I}}$, then $P^{\mathcal{C}} = \frac{1}{\alpha_{\mathcal{C}}} lnE(e^{\alpha_{\mathcal{C}} R})$ and $P^{\mathcal{I}} = \frac{1}{\alpha_{\mathcal{I}}} lnE(e^{\alpha_{\mathcal{I}} R})$.*

Following our hypotheses the player $\mathcal{C}$ may decide to accept to pay any premium (we denote this strategy by A)—i.e. he decides to believe that the premium required by the company is under his real risk even if he does not realize this—or to pay up to the amount $P^{\mathcal{C}}$ and to take the risk for higher premiums (we denote this strategy by D), so:

$$\Sigma_{\mathcal{C}} = \{A, D\}$$

Referring to player $\mathcal{I}$, a strategy corresponds to require a given premium for buying the risk R. In the real-life for each risk R a company has a finite set of n standard policies with corresponding selling prices P_i, $i = 1, \ldots, n$, with $P^{\mathcal{I}} = P_1 < P_2 < \cdots < P_n$ and $P^{\mathcal{C}} < P_n$. For sake of simplicity, we suppose to have three prices, $P^{\mathcal{I}}, \underline{P}$ and $\overline{P}$ such that $P^{\mathcal{I}} < \underline{P} \leq P^{\mathcal{C}} < \overline{P}$. The insurer does not know the risk-aversion of the customer, i.e. the price $P^{\mathcal{C}}$, however he knows that the price $P^{\mathcal{I}}$ is accepted by the customer, so he can decide to propose this price. Experience and statistical information can give

[1] As usual, in the following we denote the composed function $f \circ g$ by $f(g)$.

to the insurer the knowledge that also the higher price $\underline{P}$ is accepted by the customer (whatever his strategy), while $\overline{P}$ may be rejected by the customer (if he chooses the strategy D).

Denoting each strategy by the corresponding price we have:

$$\Sigma_{\mathcal{I}} = \{P^{\mathcal{I}}, \underline{P}, \overline{P}\}$$

According to what we said at the beginning of this section, the game may be represented as follows:

$\mathcal{I} / \mathcal{C}$	A		D	
$P^{\mathcal{I}}$	$u_{\mathcal{I}}(0)$	$u_{\mathcal{C}}(-P^{\mathcal{I}})$	$u_{\mathcal{I}}(0)$	$u_{\mathcal{C}}(-P^{\mathcal{I}})$
$\underline{P}$	$E(u_{\mathcal{I}}(\underline{P}-R))$	$u_{\mathcal{C}}(-\underline{P})$	$E(u_{\mathcal{I}}(\underline{P}-R))$	$u_{\mathcal{C}}(-\underline{P})$
$\overline{P}$	$E(u_{\mathcal{I}}(\overline{P}-R))$	$u_{\mathcal{C}}(-\overline{P})$	$u_{\mathcal{I}}(0)$	$E(u_{\mathcal{C}}(-R))$

This game has a unique NE in pure strategies corresponding to the strategy profile $(\underline{P}, D)$, with payoff $E(u_{\mathcal{I}}(\underline{P}-R))$ for player $\mathcal{I}$ and $u_{\mathcal{C}}(-\underline{P})$ for player $\mathcal{C}$.

There exists another NE in mixed strategies given by $(0,1,0)$ for player $\mathcal{I}$ and by $\left(\frac{E(u_{\mathcal{I}}(\underline{P}-R)) - u_{\mathcal{I}}(0)}{E(u_{\mathcal{I}}(\overline{P}-R)) - u_{\mathcal{I}}(0)}, \frac{E(u_{\mathcal{I}}(\overline{P}-R)) - E(u_{\mathcal{I}}(\underline{P}-R))}{E(u_{\mathcal{I}}(\overline{P}-R)) - u_{\mathcal{I}}(0)}\right)$ for player $\mathcal{C}$; note that the strategy of player $\mathcal{I}$ corresponds to the pure strategy $\underline{P}$ and that the expected value for both players is the same of the previous NE $(\underline{P}, D)$.

The maxmin strategies for the two players are $\underline{P}$ and D, respectively.

In the following example we present the game and the NE arising from a simple situation.

Example 2. *We consider a risk R and suppose that the utility functions are as in Example 1, (so $u_{\mathcal{I}}(0) = 0$); then the game is:*

$\mathcal{I} / \mathcal{C}$	A		D	
$P^{\mathcal{I}}$	0	$\frac{1}{\alpha_{\mathcal{C}}}\left(1-e^{\alpha_{\mathcal{C}}P^{\mathcal{I}}}\right)$	0	$\frac{1}{\alpha_{\mathcal{C}}}\left(1-e^{\alpha_{\mathcal{C}}P^{\mathcal{I}}}\right)$
$\underline{P}$	$\frac{1}{\alpha_{\mathcal{I}}}\left(1-e^{-\alpha_{\mathcal{I}}\underline{P}}E\left(e^{\alpha_{\mathcal{I}}R}\right)\right)$	$\frac{1}{\alpha_{\mathcal{C}}}\left(1-e^{\alpha_{\mathcal{C}}\underline{P}}\right)$	$\frac{1}{\alpha_{\mathcal{I}}}\left(1-e^{-\alpha_{\mathcal{I}}\underline{P}}E\left(e^{\alpha_{\mathcal{I}}R}\right)\right)$	$\frac{1}{\alpha_{\mathcal{C}}}\left(1-e^{\alpha_{\mathcal{C}}\underline{P}}\right)$
$\overline{P}$	$\frac{1}{\alpha_{\mathcal{I}}}\left(1-e^{-\alpha_{\mathcal{I}}\overline{P}}E\left(e^{\alpha_{\mathcal{I}}R}\right)\right)$	$\frac{1}{\alpha_{\mathcal{C}}}\left(1-e^{\alpha_{\mathcal{C}}\overline{P}}\right)$	0	$\frac{1}{\alpha_{\mathcal{C}}}\left(1-E\left(e^{\alpha_{\mathcal{C}}R}\right)\right)$

If we consider that the risk R can assume two values $\underline{R}$ and $\overline{R}$, $\underline{R} < \overline{R}$, with non-zero probabilities $q_{\underline{R}}$ and $q_{\overline{R}}$, respectively, then $E\left(e^{\alpha_{\mathcal{K}}R}\right) = e^{\alpha_{\mathcal{K}}\underline{R}}q_{\underline{R}} + e^{\alpha_{\mathcal{K}}\overline{R}}q_{\overline{R}}, \mathcal{K} = \mathcal{I}, \mathcal{C}$ and $\underline{R} < P^{\mathcal{I}} < P^{\mathcal{C}} < \overline{R}$. We can suppose that $\overline{P} < \overline{R}$.

If we suppose that $\underline{R} = 10$ and $\overline{R} = 20$ with probabilities $\frac{2}{3}$ and $\frac{1}{3}$, that $\alpha_{\mathcal{C}} = 0.5$ and $\alpha_{\mathcal{I}} = 0.1$, then $P^{\mathcal{C}} = 17.83$ and $P^{\mathcal{I}} = 14.53$; finally, we set $\underline{P} = 17$ and $\overline{P} = 19$; the resulting game is:

$\mathcal{I} / \mathcal{C}$	A		D	
$P^{\mathcal{I}}$	0	−2854.38	0	−2854.38
$\underline{P}$	2.19	−9827.54	2.19	−9827.54
$\overline{P}$	3.61	−26717.45	0	−14880.19

The NE in pure strategies $(\underline{P}, D)$ *assigns to the players the payoffs* 2.19 *and* −9827.54*, respectively; the same payoffs are provided by the NE in mixed strategies* $((0,1,0), (0.61,0.39))$.

Remark 2. *We notice that these are theoretical data, so the results of Examples 2 and 3 allow only for a methodological interpretation.*

4. Measuring skill and chance

In this section we want to investigate the measurement of skill in a stochastic situation in the light of the papers by Dreef et al. (2003, 2004). These authors define the skill in a casino game analyzing "*the relative extent to which the outcome of a game is influenced by the players, compared to the extent to which the outcome depends on the random elements involved.*" They distinguish among two kinds of random elements that they call external and internal chance moves. The former ones can be thought for example as the dealing of the cards or the spinning of roulette wheel or, in our situation, as the actual value of the risk and the latter ones as the use of mixed strategies by the opponents. As we said in the Introduction, to define the influence of the players and of the random elements, three different types of players are considered:

(i) the beginner, who has just learned the rules of the game and plays a naive strategy;

(ii) the optimal player, who has completely mastered the rules of the game and exploits his knowledge maximally in the choice of his strategy;

(iii) the fictive player, who knows in advance (i.e. before he has to decide) the realization of the random elements in the game.

Formally, we consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, two sets Σ_1 and Σ_2 and two functions $\varphi_i : \Sigma_1 \times \Sigma_2 \times \Omega \to \mathbb{R}$, $i = 1,2$. If both players are not fictive players (i.e. both of them do not know the realization of the random elements) they play the game $G = (\Sigma_1, \Sigma_2, \pi_1, \pi_2)$ where $\pi_i(\sigma_1, \sigma_2) = E(\varphi_i(\sigma_1, \sigma_2, \cdot))$, $i = 1,2$ for each $\sigma_1 \in \Sigma_1, \sigma_2 \in \Sigma_2$.

If one of the players is fictive, we have to introduce a new game. In particular, if player 1 is fictive we consider the game $G_1(S_1, \Sigma_2, v_1, \overline{\pi}_2)$ and if player 2 is fictive we consider the game $G_2(\Sigma_1, S_2, \overline{\pi}_1, v_2)$, where S_1 and S_2 are set of functions.

A strategy $f_1 \in S_1$ $[f_2 \in S_2]$ is a function $f_1 : \Sigma_2 \times \Omega \to \Sigma_1$ $[f_2 : \Sigma_1 \times \Omega \to \Sigma_2]$ such that for each $\sigma_2 \in \Sigma_2$ $[\sigma_1 \in \Sigma_1]$ the function $\omega \in \Omega \to \varphi_1(f_1(\sigma_2, \omega), \sigma_2, \omega) \in \mathbb{R}$ $[\omega \in \Omega \to \varphi_2(\sigma_1, f_2(\sigma_1, \omega), \omega) \in \mathbb{R}]$ is measurable.

The payoff functions $v_1, \overline{\pi}_2 : S_1 \times \Sigma_2 \to \mathbb{R}$ and $\overline{\pi}_1, v_2 : \Sigma_1 \times S_2 \to \mathbb{R}$ are defined as:

$$v_1(f_1, \sigma_2) = \int_\Omega \varphi_1(f_1(\sigma_2, \omega), \sigma_2, \omega) d\mathbb{P}(\omega),$$

$$\overline{\pi}_2(f_1, \sigma_2) = \int_\Omega \varphi_2(f_1(\sigma_2, \omega), \sigma_2, \omega) d\mathbb{P}(\omega),$$

$$v_2(\sigma_1, f_2) = \int_\Omega \varphi_2(\sigma_1, f_2(\sigma_1, \omega), \omega) d\mathbb{P}(\omega),$$

$$\overline{\pi}_1(\sigma_1, f_2) = \int_\Omega \varphi_1(\sigma_1, f_2(\sigma_1, \omega), \omega) d\mathbb{P}(\omega).$$

Remark 3. *When player 1 is fictive, for each pure strategy $\sigma_1 \in \Sigma_1$ we consider the strategy $f \in S_1$ such that $f(\sigma_2, \omega) = \sigma_1$ for each $\sigma_2 \in \Sigma_2$ and for each $\omega \in \Omega$, then we have $v_1(f, \sigma_2) = \pi_1(\sigma_1, \sigma_2)$ and we may identify the strategies f and σ_1. The same holds when player 2 is fictive.*

Now, we analyze the gain that each type of player may guarantee himself. If the beginner 1 [2] plays the strategy $p_1^b \in \Delta(\Sigma_1)$ $[p_2^b \in \Delta(\Sigma_2)]$ he may guarantee himself the gain:

$$U_1^b = \min_{p_2 \in \Delta(\Sigma_2)} \pi_1(p_1^b, p_2) \qquad \left[U_2^b = \min_{p_1 \in \Delta(\Sigma_1)} \pi_2(p_1, p_2^b) \right]$$

Note that $U_1^b = \min\limits_{\sigma_2 \in \Sigma_2} \pi_1(p_1^b, \sigma_2)$ and $U_2^b = \min\limits_{\sigma_1 \in \Sigma_1} \pi_2(\sigma_1, p_2^b)$. The optimal player 1 [2] may guarantee himself the gain:

$$U_1^{op} = \max_{p_1 \in \Delta(\Sigma_1)} \min_{\sigma_2 \in \Sigma_2} \pi_1(p_1, \sigma_2) \qquad \left[U_2^{op} = \max_{p_2 \in \Delta(\Sigma_2)} \min_{\sigma_1 \in \Sigma_1} \pi_2(\sigma_1, p_2) \right]$$

Finally, we consider the fictive players; as a fictive player knows the internal chance moves we do not consider mixed strategies for the opponent. We observe that for each $\sigma_1 \in \Sigma_1, \sigma_2 \in \Sigma_2$ as $\varphi_i(\sigma_1, \sigma_2, \cdot)$ belongs to $\mathcal{L}$, $i = 1, 2$, then they are measurable functions and then for each $\sigma_2 \in \Sigma_2$ $[\sigma_1 \in \Sigma_1]$ the function $\omega \in \Omega \to \max\limits_{\Sigma_1} \varphi_1(\cdot, \sigma_2, \omega) \in \mathbb{R}$ $\left[\omega \in \Omega \to \max\limits_{\Sigma_2} \varphi_2(\sigma_1, \cdot, \omega) \in \mathbb{R} \right]$ is measurable. Moreover, we define

$$\pi_1^*(\sigma_2) = \int_\Omega \max_{\Sigma_1} \varphi_1(\cdot, \sigma_2, \omega) d\mathbb{P}(\omega) \in \mathbb{R} \; \left[\pi_2^*(\sigma_1) = \int_\Omega \max_{\Sigma_2} \varphi_2(\sigma_1, \cdot, \omega) d\mathbb{P}(\omega) \in \mathbb{R} \right]$$

that represents the maximal payoff that fictive player 1 [2] may reach when player 2 [1] chooses the strategy σ_2 $[\sigma_1]$.

Then, fictive player 1 [2] may guarantee himself the amount:

$$U_1^{fi} = \min_{\sigma_2 \in \Sigma_2} \pi_1^*(\sigma_2) \qquad \left[U_2^{fi} = \min_{\sigma_1 \in \Sigma_1} \pi_2^*(\sigma_1) \right]$$

For player 1 [2], we can determine a strategy $f_1^* \in S_1$ $[f_2^* \in S_2]$ such that for each $\sigma_2 \in \Sigma_2$ $[\sigma_1 \in \Sigma_1]$ the function $f_1^*(\sigma_2, \cdot)$ $[f_2^*(\sigma_1, \cdot)]$ is measurable and $v_1(f_1^*, \sigma_2) = \pi_1^*(\sigma_2)$ $[v_2(\sigma_1, f_2^*) = \pi_2^*(\sigma_1)]$ so, f_1^* $[f_2^*]$ results to be a dominant strategy for fictive player 1 [2] and

$$U_1^{fi} = \min_{\sigma_2 \in \Sigma_2} \max_{f_1 \in S_1} v_1(f_1, \sigma_2) \qquad \left[U_2^{fi} = \min_{\sigma_1 \in \Sigma_1} \max_{f_2 \in S_2} v_2(\sigma_1, f_2) \right]$$

In order to obtain a measurable function that is a dominant strategy for fictive player 1 we suppose that $\Sigma_1 = \{s_1, \ldots, s_{n_1}\}$. For each $\sigma_2 \in \Sigma_2$ let $Q_1 = \left\{ \omega \in \Omega : \varphi_1(s_1, \sigma_2, \omega) = \max\limits_{\Sigma_1} \varphi_1(\cdot, \sigma_2, \omega) \right\}$ and iteratively let $Q_k = \left\{ \omega \in \Omega \setminus \cup_{j=1,k-1} Q_j : \right.$

$\left.\varphi_1(s_k,\sigma_2,\omega)=\max_{\Sigma_1}\varphi_1(\cdot,\sigma_2,\omega)\right\}$, $k=2,\ldots,n_1$. As the function $\omega\in\Omega\to\max_{\Sigma_1}\varphi_1(\cdot,\sigma_2,\omega)\in\mathbb{R}$ is measurable then the function $f_1^*(\sigma_2,\omega)=s_k,\forall\omega\in Q_k$, $k=1,\ldots,n_1$ is measurable and is a dominant strategy. In a similar way we can determine a measurable function that is a dominant strategy for fictive player 2.

Following the papers of Dreef et al. (2003, 2004), we define the skill as the relative influence of the players on the outcome of the game; more precisely, we take into account the *learning effect* and the *random effect*. For each player $i=1,2$, the learning effect LE_i is the difference between the expected gain of the optimal player and that of the beginner, i.e. $LE_i=U_i^{op}-U_i^b$ and the random effect (RE_i) is the difference between the expected gain of the fictive player and that of the optimal player, i.e. $RE_i=U_i^{fi}-U_i^{op}$. If the learning effect for a player is zero his skill is zero by definition, otherwise his skill is defined as:

$$skill_i=\frac{LE_i}{LE_i+RE_i}=\frac{U_i^{op}-U_i^b}{U_i^{fi}-U_i^b},\quad i=1,2 \tag{3}$$

The value of the $skill_i$, $i=1,2$ is a real number in the interval [0,1].

5. Insurance setting

Let us revert to our insurance situation;first, we have to define the behavior of the three types of players. We suppose that the beginner chooses the strategy of proposing the lowest premium $P^{\mathcal{I}}$ if he is the insurer (he has not the experience to trust the values $\underline{P}$ and $\overline{P}$) and the strategy of accepting any premium if he is the customer (he is not sure of his own capacity to evaluate the actual risk situation). According to these hypotheses the beginner insurer plays the strategy $P^{\mathcal{I}}$, so we have:

$$U_{\mathcal{I}}^b=\min_{\sigma\in\Sigma_{\mathcal{C}}}\pi_{\mathcal{I}}(P^{\mathcal{I}},\sigma)=\pi_{\mathcal{I}}(P^{\mathcal{I}},A)=\pi_{\mathcal{I}}(P^{\mathcal{I}},D)=E(u_{\mathcal{I}}(P^{\mathcal{I}}-R))=u_{\mathcal{I}}(0),$$

while the beginner customer plays the strategy A and the result is:

$$U_{\mathcal{C}}^b=\min_{\sigma\in\Sigma_{\mathcal{I}}}\pi_{\mathcal{C}}(A,\sigma)=\pi_{\mathcal{C}}(A,\overline{P})=u_{\mathcal{C}}(-\overline{P})$$

If the type of the players is optimal, then each of them plays the maxmin strategy; according to this the maxmin strategy of the insurer is $\underline{P}$ and this leads to:

$$U_{\mathcal{I}}^{op}=\max_{p\in\Delta(\Sigma_{\mathcal{I}})}\min_{\sigma\in\Sigma_{\mathcal{C}}}\pi_{\mathcal{I}}(p,\sigma)=\pi_{\mathcal{I}}(\underline{P},A)=\pi_{\mathcal{I}}(\underline{P},D)=E(u_{\mathcal{I}}(\underline{P}-R)),$$

while the maxmin strategy of the customer is D and we obtain:

$$U_{\mathcal{C}}^{op}=\max_{p\in\Delta(\Sigma_{\mathcal{C}})}\min_{\sigma\in\Sigma_{\mathcal{I}}}\pi_{\mathcal{C}}(p,\sigma)=\pi_{\mathcal{C}}(D,\overline{P})=E(u_{\mathcal{C}}(-R))$$

Finally, we analyze the case of fictive players. If the insurer is fictive, each of his strategies says the amount of premium the insurer may ask for, given the actual value of the risk and the strategy chosen by the customer. If the customer is fictive, each of his strategies says his choice, given the actual value of the risk and the amount of premium required by the insurer.

According to what we said above, a measurable dominant strategy for the fictive insurer is:

$$f^*_{\mathcal{I}}(\sigma_{\mathcal{C}},\omega)=\begin{cases}\overline{P} & \text{if } \sigma_{\mathcal{C}}=A, \text{ or } \sigma_{\mathcal{C}}=D \text{ and } R(\omega)>\underline{P}\\ \underline{P} & \text{if } \sigma_{\mathcal{C}}=D \text{ and } R(\omega)\leq\underline{P}\end{cases}$$

that corresponds to the following behavior: if the customer accepts any premium, then the choice of the insurer is to ask for the highest premium $\overline{P}$, whatever is the actual value of the risk; if the customer decides to pay no more than $P^{\mathcal{C}}$, then the insurer chooses $\underline{P}$ when the actual value of the risk is less than or equal to the premium $\underline{P}$, but chooses the premium $\overline{P}$ (so the customer does not subscribe the policy) when the actual value of the risk is greater than the premium $\underline{P}$.

We have:

$$v_{\mathcal{I}}(f^*_{\mathcal{I}},A)=E(u_{\mathcal{I}}(\overline{P}-R))$$

$$v_{\mathcal{I}}(f^*_{\mathcal{I}},D)=\int_{[0,\underline{P}]}u_{\mathcal{I}}(\underline{P}-x)dF(x)+\int_{]\underline{P},+\infty[}u_{\mathcal{I}}(0)dF(x)$$

and he can guarantee himself the amount:

$$U^{fi}_{\mathcal{I}}=\min\left\{E(u_{\mathcal{I}}(\overline{P}-R)),\int_{[0,\underline{P}]}u_{\mathcal{I}}(\underline{P}-x)dF(x)+\int_{]\underline{P},+\infty[}u_{\mathcal{I}}(0)dF(x)\right\}$$

A measurable dominant strategy for the fictive customer is:

$$f^*_{\mathcal{C}}(\sigma_{\mathcal{I}},\omega)=\begin{cases}A & \text{if } \sigma_{\mathcal{I}}=P^{\mathcal{I}} \text{or } \underline{P}, \text{ or } \sigma_{\mathcal{I}}=\overline{P} \text{ and } R(\omega)>\overline{P}\\ D & \text{if } \sigma_{\mathcal{I}}=\overline{P} \text{ and } R(\omega)\leq\overline{P}\end{cases}$$

and the corresponding behavior is: if the insurer offers the premium $P^{\mathcal{I}}$ or $\underline{P}$, then, whatever the actual value of the risk, the customer accepts the policy; if the insurer decides to ask $\overline{P}$, then the customer subscribes the policy when the actual value of the risk is greater than the premium $\overline{P}$, but he does not subscribe the policy when the actual value of the risk is less than or equal to the premium $\overline{P}$.

So we have:

$$v_{\mathcal{C}}(f^*_{\mathcal{C}},P^{\mathcal{I}})=u_{\mathcal{C}}(-P^{\mathcal{I}})$$

$$v_{\mathcal{C}}(f^*_{\mathcal{C}},\underline{P})=u_{\mathcal{C}}(-\underline{P})$$

$$v_{\mathcal{C}}(f^*_{\mathcal{C}},\overline{P})=\int_{[0,\overline{P}]}u_{\mathcal{C}}(-x)dF(x)+\int_{]\overline{P},+\infty[}u_{\mathcal{C}}(-\overline{P})dF(x)$$

and he can guarantee himself the amount:

$$U^{fi}_{\mathcal{C}}=\min\left\{u_{\mathcal{C}}(-\underline{P}),\int_{[0,\overline{P}]}u_{\mathcal{C}}(-x)dF(x)+\int_{]\overline{P},+\infty[}u_{\mathcal{C}}(-\overline{P})dF(x)\right\},$$

where we use that $u_{\mathcal{C}}(-\underline{P})\leq u_{\mathcal{C}}(-P^{\mathcal{I}})$.

Remark 4.

$$f_{\mathcal{I}}^{**}(\sigma_{\mathcal{C}},\omega)=\begin{cases}\overline{P} & \textit{if } \sigma_{\mathcal{C}}=A, \textit{ or } \sigma_{\mathcal{C}}=D, \textit{ and } R(\omega)\geq \underline{P}\\ \underline{P} & \textit{if } \sigma_{\mathcal{C}}=D, \textit{ and } R(\omega)<\underline{P}\end{cases}$$

and

$$f_{\mathcal{C}}^{**}(\sigma_{\mathcal{I}},\omega)=\begin{cases}A & \textit{if } \sigma_{\mathcal{I}}=P^{\mathcal{I}} \textit{or } \underline{P}, \textit{ or } \sigma_{\mathcal{I}}=\overline{P} \textit{ and } R(\omega)\geq \overline{P}\\ D & \textit{if } \sigma_{\mathcal{I}}=\overline{P} \textit{ and } R(\omega)<\overline{P}\end{cases}$$

are also dominant strategies.

Now, we have all the elements in order to compute the skill according to formula (3). To make clearer the above concepts we apply them to the game in Example 2.

Example 3. *We start by computing the expected gains of the two players:*

$$\begin{aligned}
U_{\mathcal{I}}^{b} &= 0\\
U_{\mathcal{C}}^{b} &= \tfrac{1}{\alpha_{\mathcal{C}}}\left(1-e^{\alpha_{\mathcal{C}}\overline{P}}\right)\\
U_{\mathcal{I}}^{op} &= \tfrac{1}{\alpha_{\mathcal{I}}}\left(1-e^{-\alpha_{\mathcal{I}}\underline{P}}\left(e^{\alpha_{\mathcal{I}}\underline{R}}q_{\underline{R}}+e^{\alpha_{\mathcal{I}}\overline{R}}q_{\overline{R}}\right)\right)\\
U_{\mathcal{C}}^{op} &= \tfrac{1}{\alpha_{\mathcal{C}}}\left(1-\left(e^{\alpha_{\mathcal{C}}\underline{R}}q_{\underline{R}}+e^{\alpha_{\mathcal{C}}\overline{R}}q_{\overline{R}}\right)\right)\\
U_{\mathcal{I}}^{fi} &= \min\left\{\tfrac{1}{\alpha_{\mathcal{I}}}\left(1-e^{-\alpha_{\mathcal{I}}\overline{P}}\left(e^{\alpha_{\mathcal{I}}\underline{R}}q_{\underline{R}}+e^{\alpha_{\mathcal{I}}\overline{R}}q_{\overline{R}}\right)\right),\tfrac{1}{\alpha_{\mathcal{I}}}\left(1-e^{-\alpha_{\mathcal{I}}\underline{P}}e^{\alpha_{\mathcal{I}}\underline{R}}\right)q_{\underline{R}}\right\}\\
U_{\mathcal{C}}^{fi} &= \min\left\{\tfrac{1}{\alpha_{\mathcal{C}}}\left(1-e^{\alpha_{\mathcal{C}}\underline{P}}\right),\tfrac{1}{\alpha_{\mathcal{C}}}\left(1-e^{\alpha_{\mathcal{C}}\underline{R}}\right)q_{\underline{R}}+\tfrac{1}{\alpha_{\mathcal{C}}}\left(1-e^{\alpha_{\mathcal{C}}\overline{P}}\right)q_{\overline{R}}\right\}
\end{aligned}$$

So, the skill of the players is:

$$skill_{\mathcal{I}}=\frac{1-e^{-\alpha_{\mathcal{I}}\underline{P}}\left(e^{\alpha_{\mathcal{I}}\underline{R}}q_{\underline{R}}+e^{\alpha_{\mathcal{I}}\overline{R}}q_{\overline{R}}\right)}{\min\left\{1-e^{-\alpha_{\mathcal{I}}\overline{P}}\left(e^{\alpha_{\mathcal{I}}\underline{R}}q_{\underline{R}}+e^{\alpha_{\mathcal{I}}\overline{R}}q_{\overline{R}}\right),\left(1-e^{-\alpha_{\mathcal{I}}\underline{P}}e^{\alpha_{\mathcal{I}}\underline{R}}\right)q_{\underline{R}}\right\}} \tag{4}$$

$$skill_{\mathcal{C}}=\frac{e^{\alpha_{\mathcal{C}}\overline{P}}-\left(e^{\alpha_{\mathcal{C}}\underline{R}}q_{\underline{R}}+e^{\alpha_{\mathcal{C}}\overline{R}}q_{\overline{R}}\right)}{e^{\alpha_{\mathcal{C}}\overline{P}}-\max\left\{e^{\alpha_{\mathcal{C}}\underline{P}},e^{\alpha_{\mathcal{C}}\underline{R}}q_{\underline{R}}+e^{\alpha_{\mathcal{C}}\overline{P}}q_{\overline{R}}\right\}} \tag{5}$$

Using the same numerical values of Example 2, we obtain $U_{\mathcal{I}}^{b}=0$, $U_{\mathcal{C}}^{b}=-26,717.45$, $U_{\mathcal{I}}^{op}=2.19$, $U_{\mathcal{C}}^{op}=-14,880.19$, $U_{\mathcal{I}}^{fi}=3.36$, $U_{\mathcal{C}}^{fi}=-9,827.54$, $skill_{\mathcal{I}}=0.652$, $skill_{\mathcal{C}}=0.701$.

According to the data used in this example and the resulting values of the skill, we gather that both the agents should be careful in the choice of their decision about the insurance contract.

Looking at the formulas (4) *and* (5) *we note that if the value of* $\underline{P}$ *increases (in the interval* $]P^{\mathcal{I}},P^{\mathcal{C}}[$*) then the value of* $skill_{\mathcal{I}}$ *increases and the value of* $skill_{\mathcal{C}}$ *weakly increases, while if the value of* $\overline{P}$ *increases (in the interval* $]P^{\mathcal{C}},\overline{R}[$*), then the value of* $skill_{\mathcal{I}}$ *decreases and the value of* $skill_{\mathcal{C}}$ *increases.*

6. Concluding remarks

In this paper we faced the problem of determining to what extent the experience of an insurer and a customer may influence his own expected payoff in subscribing an insurance policy, exploiting the model in the papers by Borm and van der Genugten (1999, 2001) and Dreef et al. (2003, 2004). We can go further analyzing the properties of the skill of the two agents.

We recall that the optimal strategy profile for the agents is the Nash equilibrium $(\underline{P}, D)$. So, the most important role in the measurement of insurer's skill is played by the choice of the premium $\underline{P}$ that the optimal insurer proposes to the customer; on the opposite for the customer's skill the decision of the optimal customer to reject the premium $\overline{P}$ larger than $P^{\mathcal{C}}$ results to be decisive. In view of this, we have a lower bound for the skill of the insurer that depends only on $\underline{P}$:

$$skill_{\mathcal{I}} \geq \frac{E(u_{\mathcal{I}}(\underline{P}-R)) - u_{\mathcal{I}}(0)}{\int_{[0,\underline{P}[} u_{\mathcal{I}}(\underline{P}-x)dF(x) + \int_{[\underline{P},+\infty[} u_{\mathcal{I}}(0)dF(x) - u_{\mathcal{I}}(0)} \tag{6}$$

and a lower bound for the skill of the customer that depends only on $\overline{P}$:

$$skill_{\mathcal{C}} \geq \frac{E(u_{\mathcal{C}}(-R)) - u_{\mathcal{C}}(-\overline{P})}{\int_{[0,\overline{P}[} u_{\mathcal{C}}(-x)dF(x) + \int_{[\overline{P},+\infty[} u_{\mathcal{C}}(-\overline{P})dF(x) - u_{\mathcal{C}}(-\overline{P})} \tag{7}$$

We give the values of the lower bound of the insurer's skill according to (6) for different values of $\underline{P}$ in the interval $[P^{\mathcal{I}}, P^{\mathcal{C}}]$, using the same values as in Example 3 for the other elements:

$\underline{P}$	$P^{\mathcal{I}}$	15.0	15.5	16.0	16.5	17.0	17.5	17.6	$P^{\mathcal{C}}$
Lower bound	0.000	0.176	0.328	0.455	0.562	0.653	0.731	0.745	0.777

Analogously, we give the values of the lower bound of the customer's skill according to (7) for different values of $\overline{P}$ in the interval $[P^{\mathcal{C}}, \overline{R}]$:

$\overline{P}$	$P^{\mathcal{C}}$	18.0	18.5	19.0	19.5	$\overline{R}$
Lower bound	0.000	0.125	0.433	0.672	0.857	1.000

Another remark considers the possibility for the fictive customer to better exploit his theoretical complete knowledge of the chance elements. The idea is that the fictive customer may decide to not subscribe the contract whenever the value of the risk is below the required premium. We avoided this approach because it is necessary to allow the customer a third strategy that corresponds to "not to insure the risk"; this behavior does not fit the setting we choose, as in this case the customer does not enter in interaction with the insurer, so the game does not take place.

Another possible development is to perform an analysis on a stochastic sample or to design suitable experiments in order to obtain information on the data of the problem, e.g. the values of proposed or accepted premium, or on the strategic behavior of agents that may be considered as beginners or optimal players, with the aim of obtaining the real evaluation of the skill using our method.

Acknowledgment The authors gratefully acknowledge two anonymous referees for their useful suggestions.

References

Borm, P. and van der Genugten, B. (1999). On the Exploitation of Casino Games: How to Distinguish between Games of Chance and Games of Skill? In Patrone, F., Garcia-Jurado, I. and Tijs, S. (eds.), *Game Practice: Contributions from Applied Game Theory*. Dordrecht, Kluwer Academic Publisher, 19–33.

Borm, P. and van der Genugten, B. (2001). On a Relative Measure of Skill for Games with Chance Elements. *TOP*, 9, 91–114.

Dreef, M., Borm, P. and van der Genugten, B. (2003). A New Relative Skill Measure for Games with Chance Elements. *Managerial and Decision Economics*, 25, 255–264.

Dreef, M., Borm, P. and van der Genugten, B. (2004). Measuring Skill in Games: Several Approaches Discussed. *Mathematical Methods of Operations Research*, 59, 375–391.

Fudenberg, D. and Tirole, J. (1991). *Game Theory*. Cambridge, MIT Press.

Goovaerts, M. J., De Vylder, F. and Haezendonck, J. (1984). *Insurance Premiums*. Amsterdam, North-Holland.

Larkey, P., Kadane, J. B., Austin, R. and Zamir, S. (1997). Skill in Games. *Management Science*, 43, 596–609.